Signals and Systems Analysis

Course Code: (ECEG-2121)

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Course title: Signals and Systems Analysis
ECTS (Credit): 6hr (4hr)
Pre-requisite: Applied Math III
Academic Year: 2019/20

Course Description

- Classification of signals and their representation as function **of time and frequency domain;** Fourier Series & Fourier transform of periodic and non-periodic signals and their properties; Laplace transform of signals:- properties and their application; representation of systems:- classification, convolution integral and system modeling using impulse response & transfer function;
- **Analysis of LTI systems:-** using differential equations solution, frequency response; discrete time signals and systems:- sampled data (sequences), convolution sum, Z-transform, system analyses in Z-domain, system function and their realization.

Course Objectives

Students who **successfully complete** the course will be able to:

- **Understand and apply** the representation different signals,
- Classification characterization different signals
- analysis of signals and systems in time and frequency domain
- properties and their application; representation of systems
- Representation of frequency response; discrete and continuous time signals and systems

Text Book:

- Signals and Systems, Second Edition, **Simon Haykin** and Barry Van Veen, John, Wiley & Sons, 2003

References:

- Signals and systems, **A.P. Oppenheim,** A.S. Willsky, I.T. Young, 2001
- Roberts: Signals and Systems: Analysis using Transform Methods and MATLAB, MJ, International Edition, McGraw Hill, 2003.
- Philip Denbigh: System Analysis and Signal, 1988
- ******** AMU – AmiT Digital Library Sources (additionally you can referee)
    - drs.amu.edu.et
    - ils.amu.edu.et

Evaluation and assessment

Evaluation Description		Weight/100%	Date
Lecture	Test I	15%	
	Assignment I	10%	
	Test II	20%	
	Quiz	5%	
	Final Exam	45%	

Chapters

Chapter-1 Introduction to Signal and System

Chapter-2 Time Domain Representations for LTIS

Chapter-3 Fourier Representations for Signals

Chapter-4 Laplace Transform

Chapter-5 Z-transform

Chapter 1

Introduction to Signals and Systems

1.1 Introduction to Signal and System

- Signal **and** System analysis is a fundamental course for **electrical, electronics computer engineering filed.**

Definition of terms

- **Mathematical model**
- **Model**
- **Signal**
- **System**

Cont.

Mathematical Model: is representation of a **system or real world** problem using mathematical expression(language). **and**

- The process of developing mathematical model is known as **mathematical modeling.**

Model: is a simplified form of representation of a **system or real world** events using :

- **Mathematical equation**
- **Graphs**
- **Drawing**
- **Prototype**

Cont.

Signal: is an abstraction of any measurable quantity that is a function of one or more independent variables such as **time(t)** and **space(x,y,z)which intended conveys information.** Or

- **signal:** is a set of information of data.

A signal: is formally defined as a function of ***one or more variables***, which carries information on the nature of physical phenomena.

Example of Signal

Electrical signal

- **Voltage v(t) and Current I(t)**

Mechanical signal

- **Sound signal**

Cont …

Light signal

- Image

When a signal depends on ***a single variable***, the signal is said to be ***one-dimensional (single valued) signal.***

- A **speech signal:** is an example of **a one dimensional** signal whose amplitude varies with time, depending on the spoken word and who speaks it.

Cont …

When **the function depends** on **two or more variables**, the signal is said to be ***multidimensional signal.***

- ***An image*** is an example of ***a two dimensional signal***, with the horizontal and vertical coordinates of the image representing the two dimension.

- In this course we focus on signals involving ***a single independent variable.***

System

A system is formally defined as an entity that **manipulates one or more signals** to accomplish a function, thereby **yielding new signals. or**

System: is defined as **interconnection** of **one or more networks** in **a designed manner** to perform desired task.

- The interaction between a system its associated signals is illustrated schematically in **figure 1.**

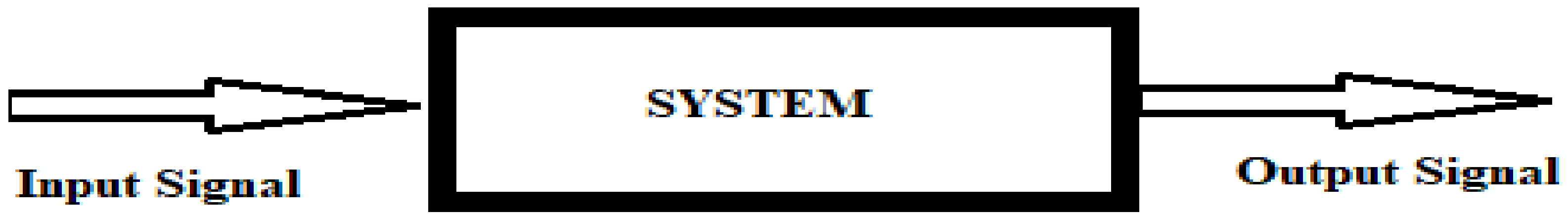

FIGURE 1: Block diagram representation of a system

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The descriptions of the input and output signals naturally depends on the intended application of the system:

- In **an automatic speaker recognition system**, the **input signal** is a speech (voice) signal, the system is a computer and the **output signal** is the **identity** of the speaker

- In **a communication system**, the input signal could be a speech signal or computer data, the system itself is made up of the combination of a ***transmitter, channel and receiver*** and the output signal is an estimate of **the original message signal.**

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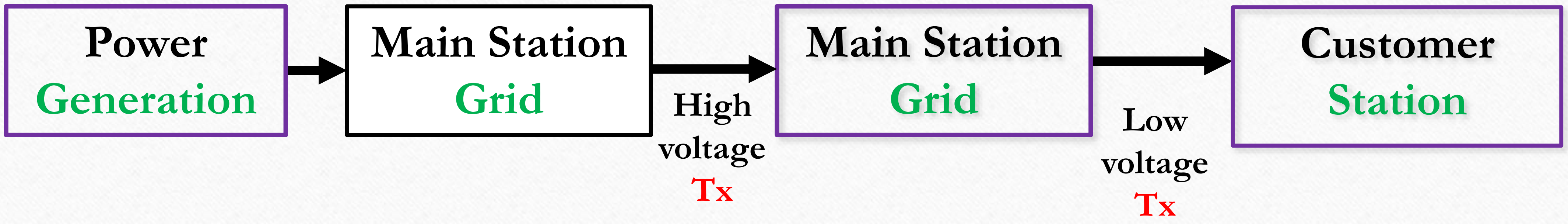
Electronic Communication System

Electrical Power System



1.2 Classification of Signals

In this course we will restrict our attention to ***one-dimensional signals*** defined as ***single valued functions of time.***

"single valued" means that for every instant of time there is a unique value of the function.

- This value may be a real number, in which case we speak of ***a real valued signal,*** or
- it may be a complex number, in which case we speak of ***a complex-valued signal.***

- ***In either case,*** the independent variable, ***namely time, is real valued.***

Cont …

- We may identify **five methods of classifying signals** based on different features.

1. **Continuous –Time and Discrete-Time Signals**
2. **Even and Odd Signals**
3. **Period Signal and Non-Periodic Signal**
4. **Deterministic Signals and Random Signals**
5. **Causal and Non-causal Signal**
6. **Energy Signals and Power Signals**

1. Continuous –Time and Discrete-Time Signals

Continuous-time signals

- A signal ***x(t)*** is said to be **a continuous-time** signal if it is defined for all time **'t'.**

- **Figure 2(a),** represents an example of **a continuous-time signal** whose amplitude or values varies continuously with time.
- Continuous-time signals arise naturally when a physical waveform such as **an acoustic (audio)** wave or **light wave** is converted in to an **electrical signal.**

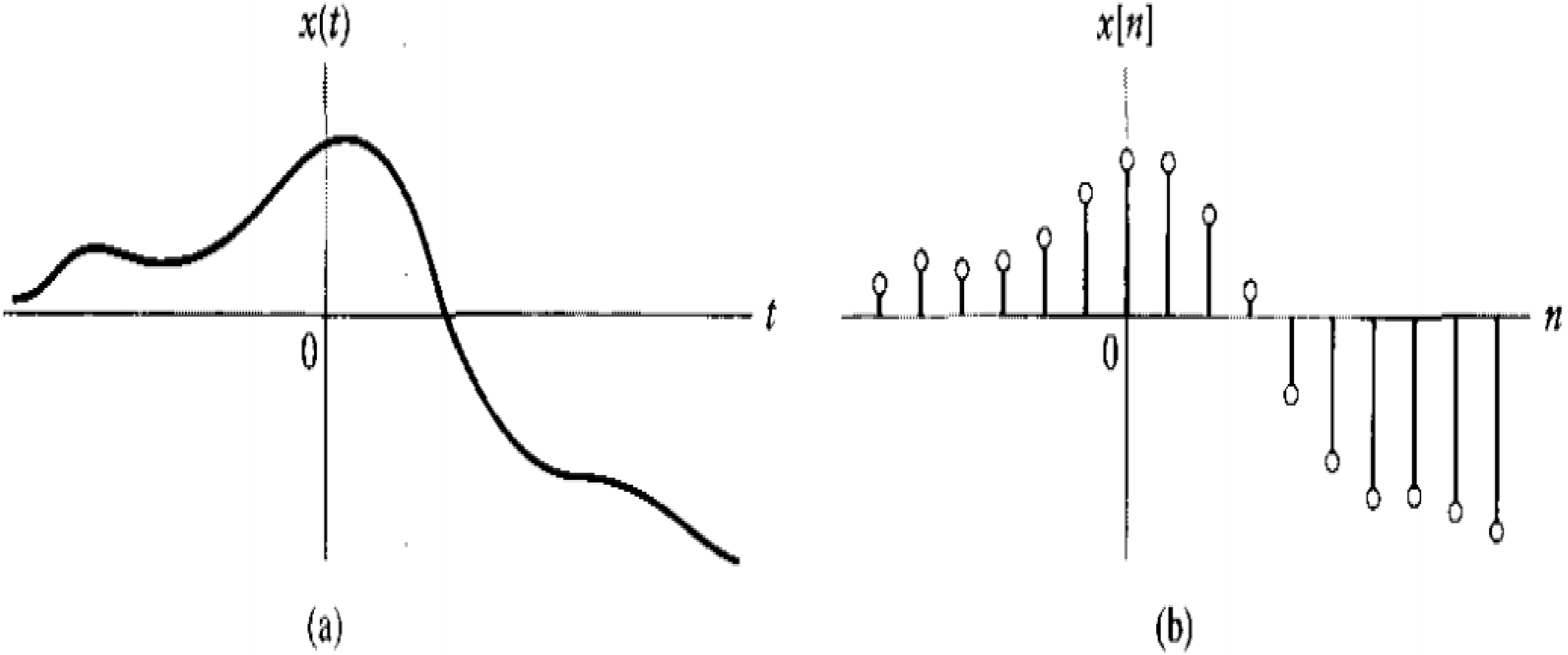


Figure 2: (a) Continuous-time signal *x(t)* (b) Representation of *x(t)* as a discrete-time signal *x[n]*

Cont …

- The conversion is effected by means of a transducer; examples include the **microphone,** which converts sound pressure variations in to corresponding ***voltage or current variations,*** and
- **The photocell,** which does the same for ***light-intensity variations.***

Discrete-time

- In the case of DT-signal **x[n]** is defined only at discrete instant of time.
- Thus, in this case, the independent variable has discrete values only, **which are usually uniformly** spaced, see **Figure 2(b) is DT-Signal.**
- A discrete-time signal is often derived from a continuous-time signal by ***sampling it at a uniform rete.***

Cont …

- **Let T denote the sampling period and 'n' denote an integer that may assume positive and negative values.**
- Sampling a continuous-time signal $\boldsymbol{x(t)}$ at time $\boldsymbol{t = nT}$ yields a sample of values $\boldsymbol{x(nT)}$.
- For convenience of presentation, we write:

$$\boldsymbol{x[n] = x(nT)}; \qquad \boldsymbol{n = 0, \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \cdots}$$

- Thus, a discrete-time signal is represented by the sequence numbers

$$\cdots, \boldsymbol{x[-2], x[-1], x[0], x[1], x[2]}, \cdots$$

Cont …

- Such a sequence of numbers is referred to **as a time series**, written as:

$$\{x[n], n = 0, \pm 1, \pm 2, \cdots\} \qquad or\ simply\ x[n]$$

the latter natation is used through out this course.

- Throughout this course, we use the symbol:
  - $'t'$ to denote time for a **CT-signal** and the symbol $'n'$ for a **DT-signal.**
  - **Parenthesis (.)** are used to denote **continuous-valued quantities** while **bracket [.]** are used to denote **discrete-valued quantities.**

2. Even and Odd Signals

- **A continuous-time signal $x(t)$** is said to be an even signal if it satisfies the condition:

$$x(-t) = x(t) \qquad for\ all\ 't'$$

- **A continuous-time signal $x(t)$** is said to be an odd signal if it satisfies the condition:

$$x(-t) = -x(t) \qquad for\ all\ 't'$$

- In other word;
  - **Even signals are symmetric** about the **vertical axis or time origin,** where as **odd signals are anti-symmetric (asymmetric) about the time origin.**

Notice: ***Similar remarks apply to discrete-time signals.***

Example 1: Develop the **Even**/**Odd** decomposition of a general signal $\boldsymbol{x(t)}$ by applying the definition of ***even and odd*** function:

Solution: Let the signal $x(t)$ be expressed as the sum of two components $\boldsymbol{x_e(t)}$ and $\boldsymbol{x_o(t)}$ as follows:

$$\boldsymbol{x(t) = x_e(t) + x_o(t)}$$

- Define $\boldsymbol{x_e(t)}$ to be even and $\boldsymbol{x_o(t)}$ to be odd; that is:

$$\boldsymbol{x_e(-t) = x_e(t)} \qquad \text{and} \qquad \boldsymbol{x_o(-t) = -x_o(t)}$$

- Putting $t = -t$; in the expression for $\boldsymbol{x(t)}$; we may then write

$$x(-t) = x_e(-t) + x_o(-t)$$

$$\boldsymbol{x(-t) = x_e(t) - x_o(t)}$$

Cont …

- Solving for $x_e(t)$ *and* $x_o(t)$, we thus obtain:

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad \textbf{and} \qquad x_o(t) = \frac{x(t) - x(-t)}{2}$$

- The above definition of ***even and odd*** signals assume that ***the signals are real valued.***
- In the case of ***complex-valued signal***, we may speak of ***conjugate symmetry.***
- A complex-valued signal is said to be **conjugate symmetric** if it satisfies the condition:

$$x(-t) = x^*(t)$$

Where the **"asterisk"** denotes **complex conjugate**.

Cont …

- Let;

$$\boldsymbol{x(t) = a(t) + jb(t)}$$

Where:

- ➢ $\boldsymbol{a(t)}$ ***and*** $\boldsymbol{b(t)}$ are the **real and imaginary** part of $\boldsymbol{x(t)}$ respectively
- ➢ ***j is the square root of -1***

- The complex conjugate of $\boldsymbol{x(t)}$ is:

$$\boldsymbol{x^*(t) = a(t) - jb(t)}$$

- **From the previous equation it follows that:** a complex values signal $\boldsymbol{x(t)}$ is conjugate symmetric if **its real part is even** and **its imaginary part is odd.**
- ***Notice:*** similar remark applies to ***a discrete-time signal.***

Example 2: consider the pair of signals shown in ***figure 3.*** Which of **these signals is even and which one is odd?**

Example 3: The signals $x_1(t)$ **and** $x_2(t)$ shown in ***figure 3 (a) and (b)*** constitute the **real and imaginary parts** of **a complex-valued signal** $x(t)$.

What form of **symmetry does** $x(t)$ have?

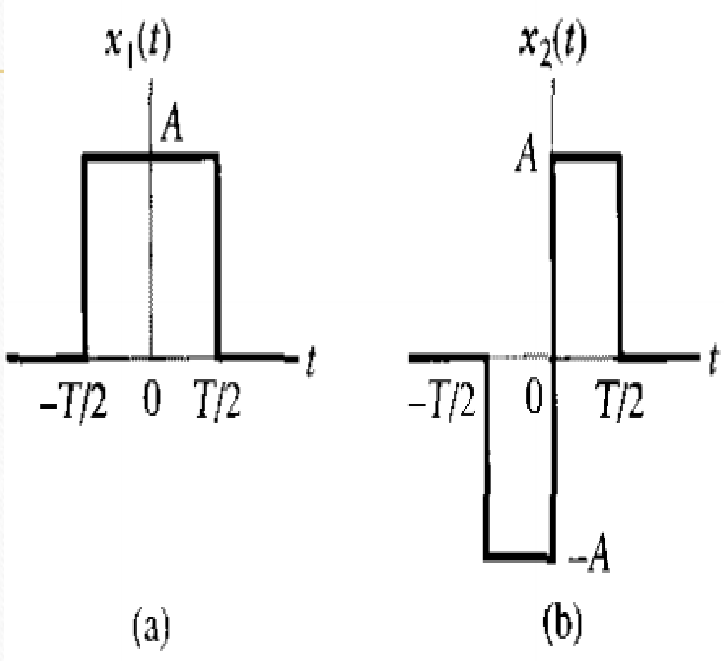


Figure 3: Examples of Continuous-Time Signal

3. Periodic Signal and Non-Periodic Signal

- **A periodic signal $x(t)$** is a function that satisfies the condition:

$$x(t) = x(t+T) \qquad for\ all\ t$$

Where: **T is a positive constant**

- Clearly, if this condition is satisfied for $\mathbf{T} = \boldsymbol{T_o}$, say, then it is also satisfied for $\boldsymbol{T = 2T_o, 3T_o, 4T_o, \cdots}$.
- The smallest value of T that satisfies the above equation is called the **fundamental period of $x(t)$.**
- The reciprocal of the **fundamental period T** is called the **fundamental frequency(f)** of the periodic signal $x(t)$;

Cont …

- We thus formally write;

$$f = \frac{1}{T}$$

- The frequency f is measured in **hertz** $(\boldsymbol{Hz})$ or **cycles per second.**
- The **angular frequency,** measured in **radians per second**, is defined by:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

- Since there are **2π *radians in one complete cycle.***
- To simplify the terminology, **ω is often referred to simply as frequency.**

Cont …

- Any ***signal*** $\boldsymbol{x(t)}$ for which there is **no values of T** to satisfy the condition of periodic signal is called ***aperiodic or non-periodic signal.***

- ***Figures 4 (a) and (b)*** presents examples of **periodic and non-periodic *signals,*** respectively.

- The periodic signal shown here represents a square wave of amplitude $\boldsymbol{A = 1}$ **and period T** and the non-periodic signal represents a rectangular pulse of **amplitude A and duration $\boldsymbol{T_1}$.**

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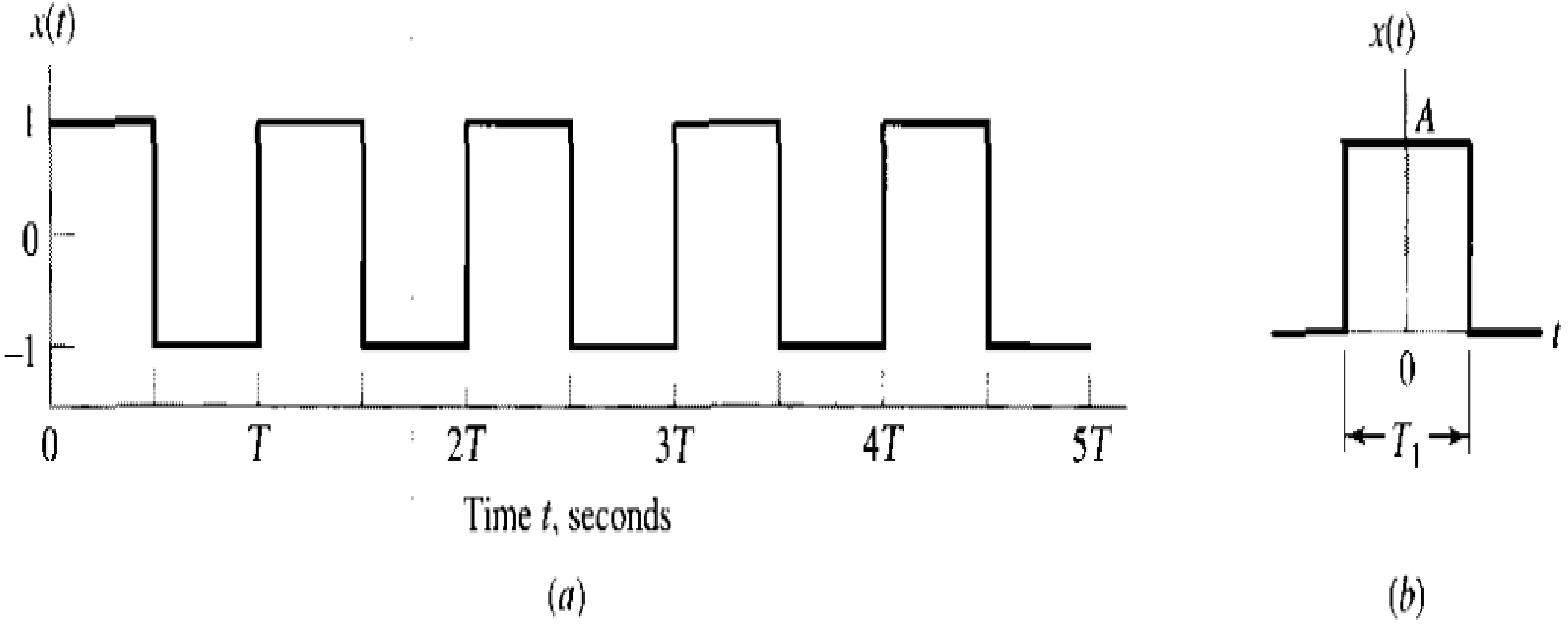


Figure 4: (a) Square wave with amplitude A = 1, and period T = 0.2 s (b) Rectangular pulse of amplitude A and duration T1

Example 4:

- ***Figure 5*** shows a triangular wave. What is the ***fundamental frequency*** of this wave? Express the fundamental frequency in units of **Hz or rad/s.**

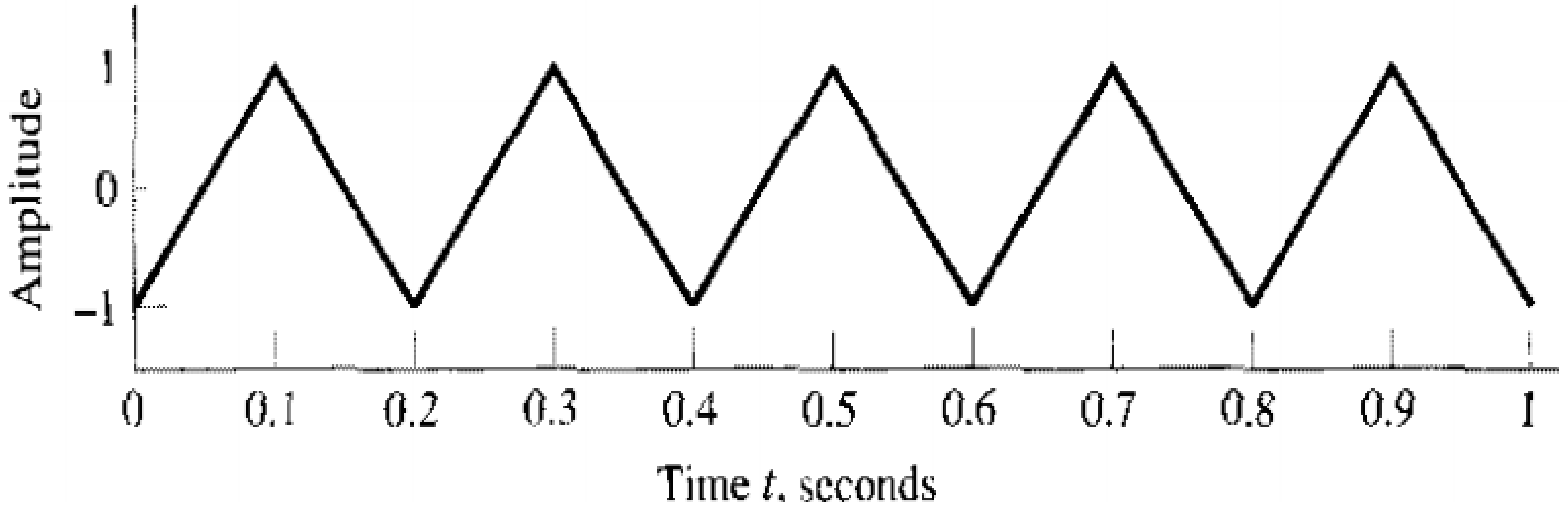


Figure 5: Triangular Wavealtering between - 1 and + 1 with fundamental period of 0.2 second

- We next consider the case of ***discrete-time signals.***
- A discrete time signal $\boldsymbol{x}[\boldsymbol{n}]$ is said to be periodic if it satisfies the condition:

$$\boldsymbol{x}[\boldsymbol{n}] = \boldsymbol{x}[\boldsymbol{n}+\boldsymbol{N}] \qquad \boldsymbol{for\ all\ integers\ n}$$

Where N; is a positive integer.

- The smallest value of ***integer N*** for which the above equation is satisfied is called fundamental period of ***the discrete-time signal*** $\boldsymbol{x}[\boldsymbol{n}]$.
- The fundamental angular frequency, or simply fundamental frequency of $x[n]$ is defined by:

$$\Omega = \frac{2\pi}{\boldsymbol{N}} \ \textbf{(in radians)}$$

Cont …

- The differences between the defining equations for continuous-time and discrete-time periodic signals should be carefully noted.
  - The earlier equation applies to a periodic continuous-time signal whose fundamental period ***T has any positive value.***
  - On the other hand, the later equation applies to ***a periodic discrete-time signal*** whose fundamental period **N can only assume a positive integer value.**
- Two examples of **discrete-time signals** are shown in ***Figure 6 (a)*** and **(b)** below. The signal in **Fig. 6(a)** is periodic where as that of **Fig. 6(b)** is aperiodic.

Example 5: What is the fundamental frequency of the discrete –time square wave shown in **Fig. 6(a)**

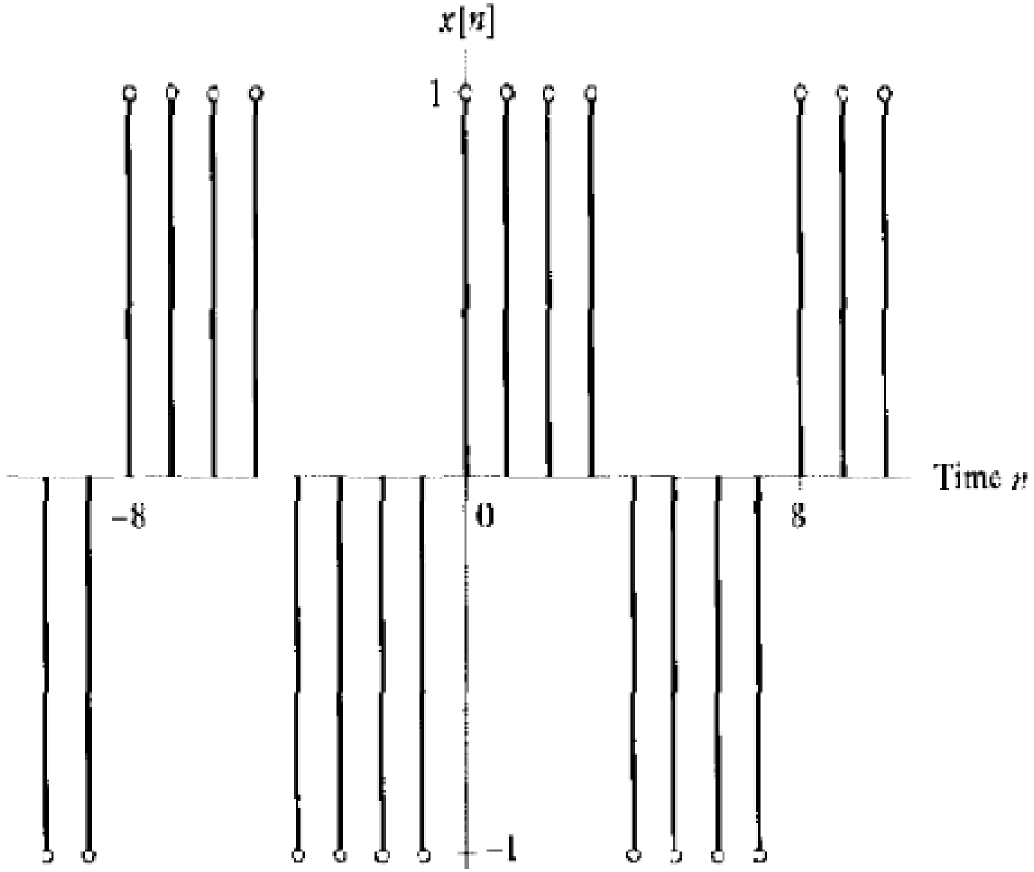


Figure 6(a): Discrete-Time Square Wavealternating between - 1 and + 1

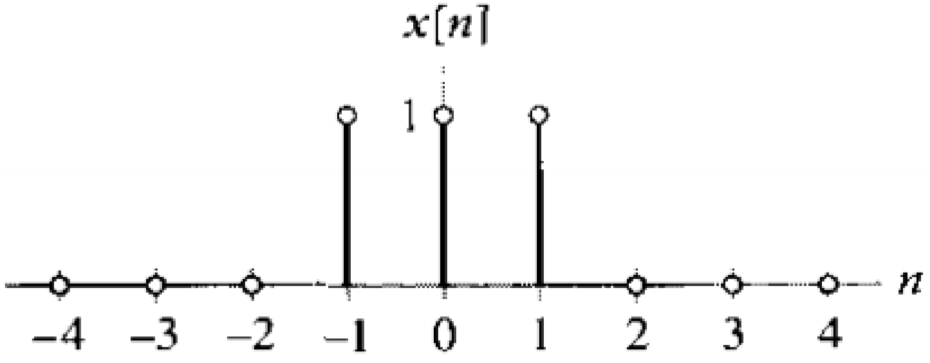


Figure 6(b): Aperiodic Discrete-Time Signal consisting of three non-zero samples

4. Deterministic Signals and Random Signals

- **A deterministic signal** is a signal about which there is **no uncertainty** with respect to its value at any time.
- Accordingly, we find that deterministic signals may be ***modeled as completely specified function of time.***
  - The square wave shown in ***figure 4(a)*** and the rectangular pulse shown in shown in ***figure 4(b)*** are deterministic signals, and so are the signals shown in ***figure 6(a) and (b).***
- ***A random signal*** is a signal about which there is ***uncertainty*** before its actual occurrence.

Cont …

- Such a signal may be viewed as belonging to an **ensemble(collective) or group** of signals, with each signal in the ensemble having a different wave form.
- ***Moreover,*** each signal within the ensemble has certain probability of occurrence.
- The ensemble of such signals is referred to as ***a random process.***
- The **"noise"** generated in the amplifier of ***a radio or television receiver*** is an example of ***a random signal.*** Its amplitude fluctuates between positive and negative values in a completely random fashion.

5. Energy Signals and Power Signals

- From examples provided so far, we see that signals may represent a broad variety of phenomena.
- In many, but not all, applications, the signals we consider are directly related to physical quantities capturing ***power and energy*** in a physical system.
- For example, if $\boldsymbol{v(t)}$ **and** $\boldsymbol{i(t)}$ are, respectively, the voltage and current across a resistor with resistance R, then the ***instantaneous power*** dissipated in this resistor is defined by:

$$p(t) = v(t) * i(t) = \frac{1}{R} v^2(t) = Ri^2(t)$$

Cont …

- In both cases, the instantaneous ***power*** $p(t)$ is proportional to the squared amplitude of **the signal**. Further more, for a resistance ***R of 1Ω***,
- the ***two equations*** take on the same mathematical form.
- The **total energy** expended over the time **interval** $t_1 \leq t \leq t_2$ is:

$$E(t) = \int_{t_1}^{t_2} p(t)d_t = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)d_t$$

- The average power over this time interval is:

$$p_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)d_t = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t)d_t$$

Cont ...

- In signal analysis, it is customary to define power in terms of **a 1Ω resistor**, so that, regardless of whether a given **signal $x(t)$** represents ***a voltage or a current,*** we may express the instantaneous power of the signal as:

$$p(t) = x^2(t)$$

- Based on this convention, we define the **total energy of the continuous-time signal $x(t)$ as:**

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t) d_t = \int_{-\infty}^{\infty} x^2(t)\, d_t$$

Cont ...

- And its **average power** as:

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) d_t$$

- From the last equation, we readily see that the ***average power of a periodic signal is*** $x(t)$ of **fundamental period T** is given by:

$$P_{av} = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) d_t$$

Cont …

- The **square root** of the **average power p_{av}** is called the **root meal-square (rms) value of the signal $x(t)$.**
- **In the case of discrete-time signal $x[n]$,** the integrals are replaced by corresponding sums. Thus the total energy of $x[n]$ is defined by:

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

And its average power is defined by:

$$P_{av} = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} x^2[n]$$

Cont …

- Here again, the average power in ***a periodic signal*** $\boldsymbol{x[n]}$ with fundamental **period N** is given by:

$$P_{av} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

- A signal is referred to as **an energy signal**, if and only if the total energy of the signal satisfies the condition:

$$0 < E < \infty$$

- On the other hand, it is referred to **as a power signal**, if and only if the average power of the signal satisfies the condition:

$$0 < P_{av} < \infty$$

Cont …

- The **energy and power classifications** of signals are **mutually exclusive.**
- **In particular:**
  - An energy signal has **zero average power,** whereas **a power signal** has **infinite energy.**
  - **Periodic signals and Random signals** are usually viewed as **power signals,** where as signals that are both **deterministic and non-periodic** are ***energy signals.***

Example 6:

- What is the total energy of:
  a) the rectangular pulse shown in figure 7(a)?
  b) the discrete-time signal shown in figure 7(b)?

- What is the average power of:
  c) the square wave shown in figure 7(c)?
  d) the triangular wave shown in figure 7(d)?
  e) The periodic discrete-time signal shown in figure 7(e)?

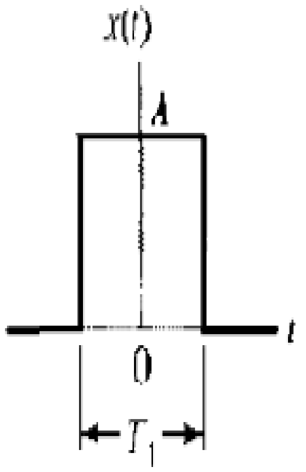


Figure 7(a) Rectangular Pulse

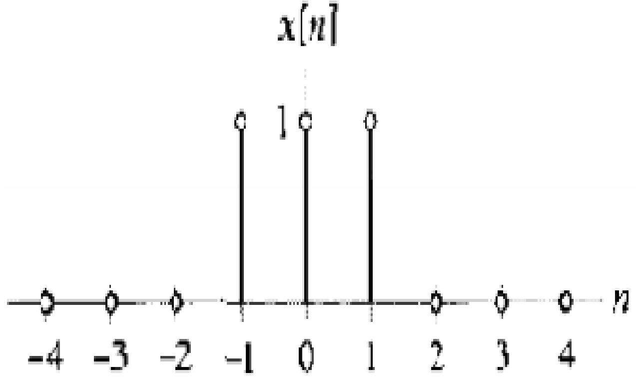


Figure 7(b) Aperiodic Discrete-Time Signal

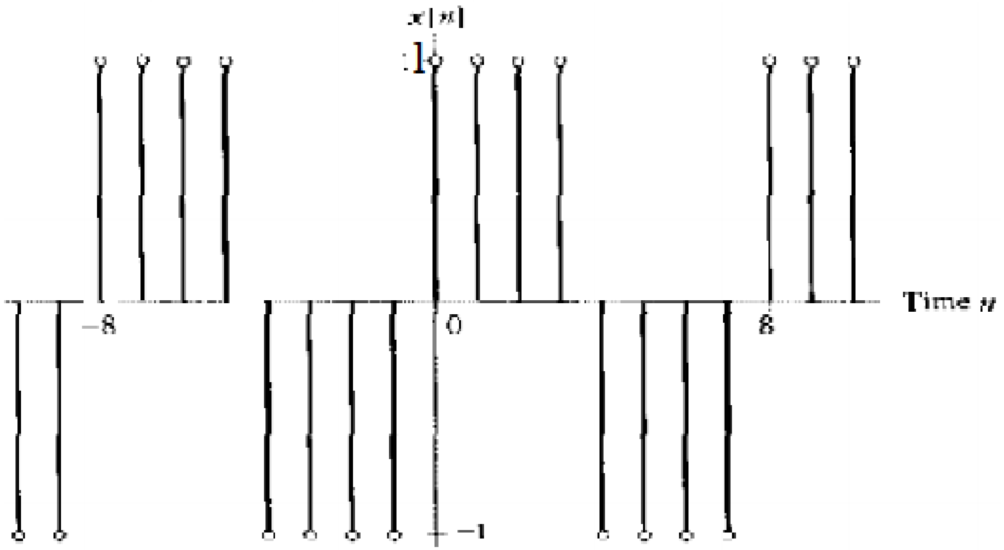


Figure7(e) Discrete-Time Square Wave

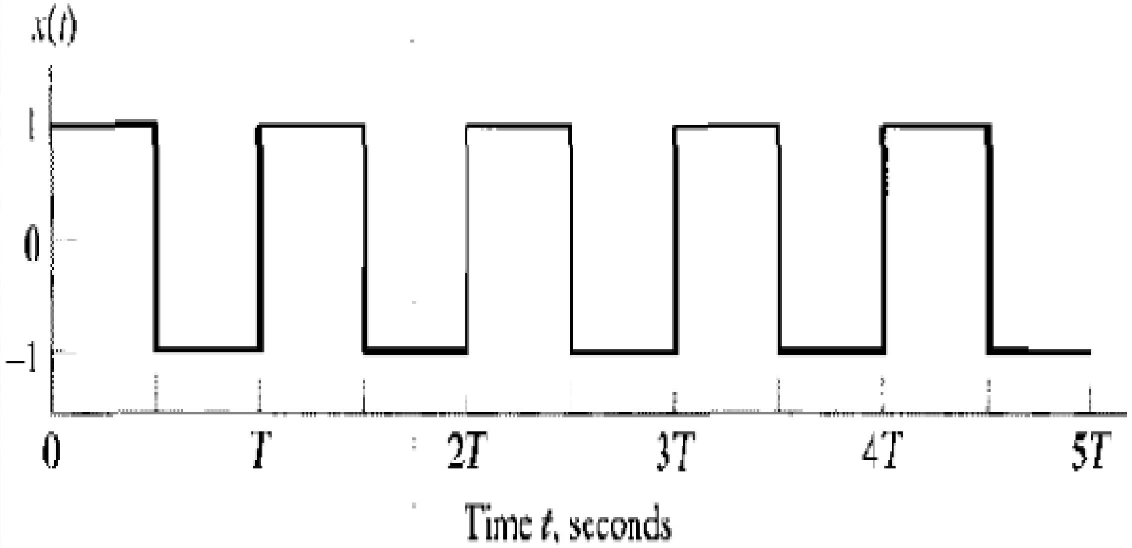


Figure 7(c) Continuous-Time Square Wave

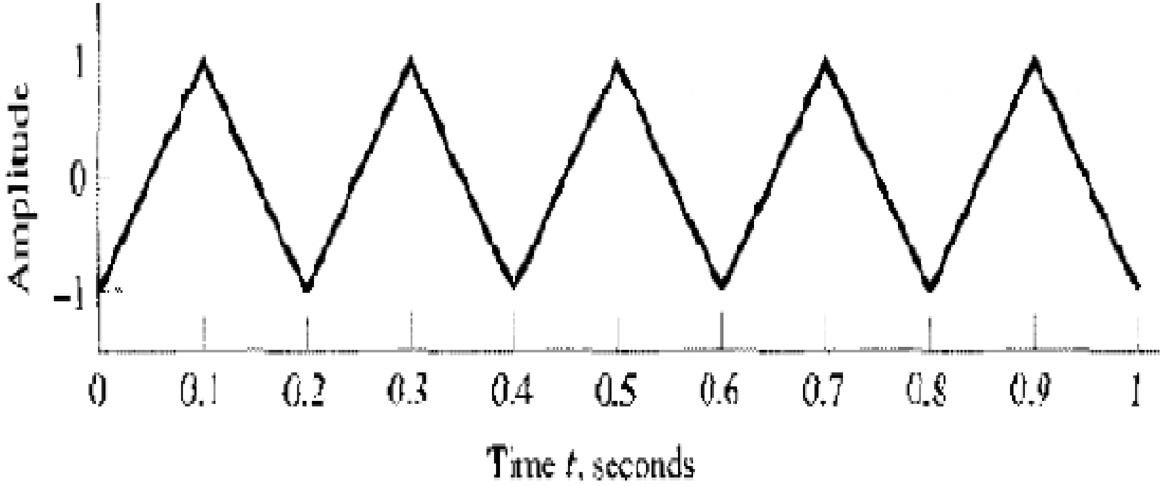


Figure 7(d) Triangular Wave

1.2 Basic operations on Signals

- An issue of fundamental importance in the study of **signals and systems** is the use of ***systems to process or manipulate signals.***
- This issue usually involves a combination of ***some basic operations.***
- In particular, **we may identify a two classes of operations;** described as:

1) **Operations performed on dependent variables:**
   - Amplitude scaling, Addition/Subtraction, Multiplication, Differentiation, and Integration
2) **Operations performed on independent variables:**
   - Time scaling, Reflection, and Time shifting

1. Operations performed on dependent variables

Amplitude Scaling:

- Let $x(t)$ denotes a continuous time signal.
- The signal $y(t)$ resulting from amplitude scaling applied to $x(t)$ is defined by:

$$y(t) = cx(t)$$

Where: c is the scaling factor.

- According to the above equation, the value of $y(t)$ is obtained by multiplying the corresponding values of $x(t)$ by the **scalar c.**

Cont …

- A physical example of a device that **performs amplitude scaling is an electronic Amplifier.**
- A resistor also performs amplitude scaling when $\boldsymbol{x(t)}$ is a current, c is the resistance and $y(t)$ is the output voltage.
- In a similar manner, for discrete-time signals we write:

$$\boldsymbol{y[n] = cx[n]}$$

Addition signals:

- Let $x_1(t)$ and $x_2(t)$ is defined by:

$$\boldsymbol{y(t) = x_1(t) + x_2(t)}$$

Cont …

- A physical example of a device that adds signals is **an audio mixer** which ***combines music and voice signals.***
- In a similar manner, for discrete time signals we write:

$$y[n] = x_1[n] + x_2[n]$$

Multiplication of signals:

- Let $x_1(t)$ and $x_2(t)$ denote a pair of ***continuous-time signals.***
- The *signal* $y(t)$ resulting from the multiplication of $x_1(t)$ by $x_2(t)$ is defined by:

$$y(t) = x_1(t)x_2(t)$$

Cont …

- That is, for each prescribed time 't' the value of $y(t)$ is given by the product of the corresponding values of $x_1(t)$ and $x_2(t)$.
- A physical example of $y(t)$ is an **AM radio signal**, in which $x_1(t)$ consists of **an audio signal plus a DC component**, and $\boldsymbol{x_2(t)}$ ***consists of a sinusoidal signal called a carrier wave.***
- In a similar manner, for discrete-time signals we write:

$$\boldsymbol{y[n] = x_1[n]x_2[n]}$$

Cont …

Differentiation of signal:

- Let $x(t)$ denote a continuous-time signal.
- The derivation of $x(t)$ with respect to time is defined by:

$$\boldsymbol{y(t) = \frac{d}{dt}x(t)}$$

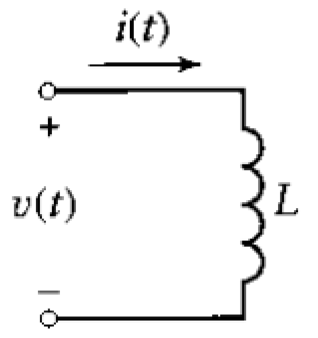


Figure 8: Inductor with current i(t), inducing voltage v(t) across its terminals

- For example, **an inductor performs differentiation.**
- Let $i(t)$ denote the current flowing through an inductor of inductance L, as shown in figure 8.
- The **voltage $v(t)$** developed across the inductor is defined by:

$$\boldsymbol{v(t) = L\frac{d}{dt}i(t)}$$

Cont …

Integration of signal:

- Let $x(t)$ denote a continuous-time signal.
- The integral of $x(t)$ with respect to time 't' is defined by:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d_\tau$$

Where: **τ is the integration variable.**

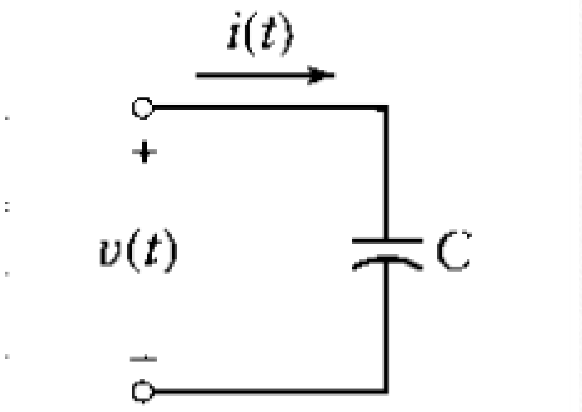


Figure 9: Capacitor with v(t) across its terminals, inducing current i(t)

- For example, a capacitor performs integration. Let $i(t)$ denote the current flowing through a capacitor of capacitance C, as shown in ***figure 9.***
- The voltage $v(t)$ developed across the capacitor is defined by;

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau) d_\tau$$

2. Signal operations performed on independent variables

Time Scaling:

- Let $x(t)$ denote a continuous time signal.
- The signal $y(t)$ obtained by scaling the independent variable, **time t**, by ***a factor 'a'*** is defined by:

$$y(t) = x(at)$$

  - If $a > 1$, the **signal** $y(t)$ is ***a compressed version of*** $x(t)$.
  - If $0 < a < 1$, the **signal** $y(t)$ is ***a expanded (stretched) version of*** $x(t)$.

- The two operations are illustrated in ***figure 10 below.***

Cont …

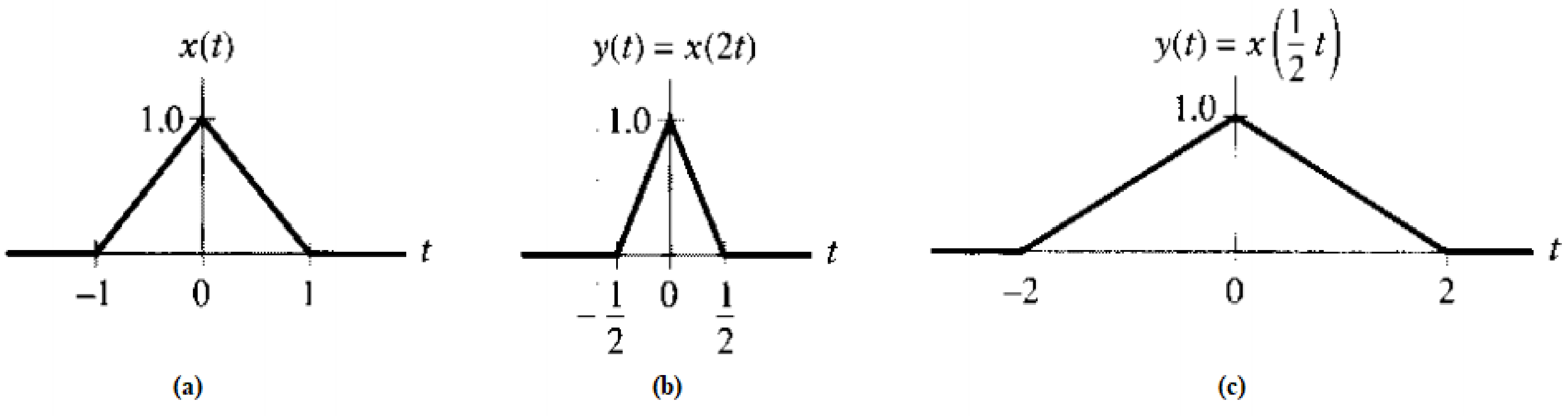


Figure 10: Time Scaling operation: (a) Continuous-Time Signal x(t), (b) compressed version of x(t) by a factor of 2, (c) expanded version of x(t) by a factor of 2.

- In discrete-time case, we write:

$$y[n] = x[kn], \qquad k > 0$$

Which is defined only for ***integer values of k.***

- If **k>1**, then some values of the **discrete-time signal $y[n]$ are lost**, as illustrated in **figure 11 for k = 2.**

Cont …

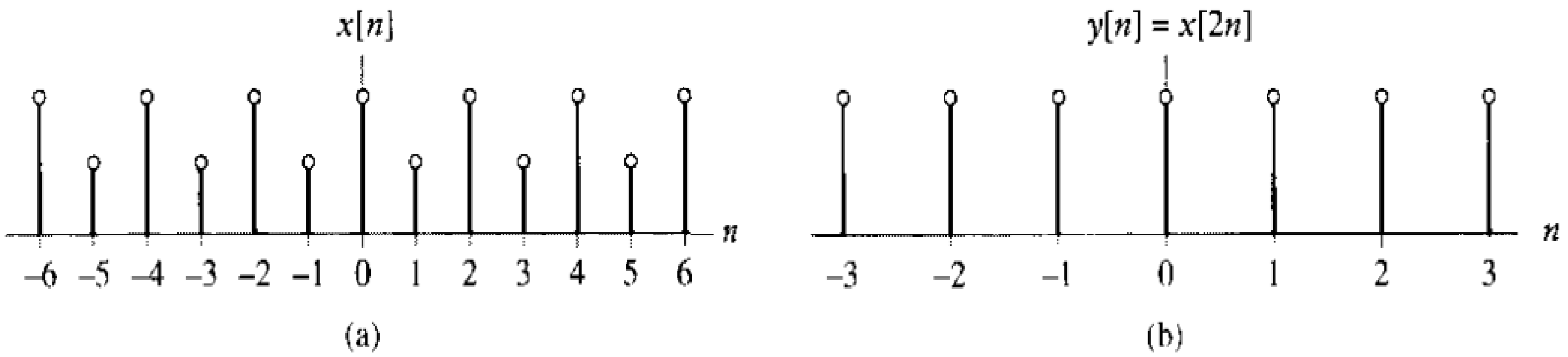


Figure 10: Effect of time scaling on a discrete-time signal: (a) discrete-time signal x[n], and (b) compressed version of x[n] by a factor of 2, with some values of the original x[n] lost as a result of the compression.

Reflection:

- Let $\boldsymbol{x(t)}$ represents a continuous-time signal.
- Let $\boldsymbol{y(t)}$ denote the signal obtained by replacing time **'t' by '-t',** as shown by:

$$\boldsymbol{y(t) = x(-t)}$$

- The **signal $\boldsymbol{y(t)}$** represents a reflected version of $\boldsymbol{x(t)}$ about the amplitude axis.

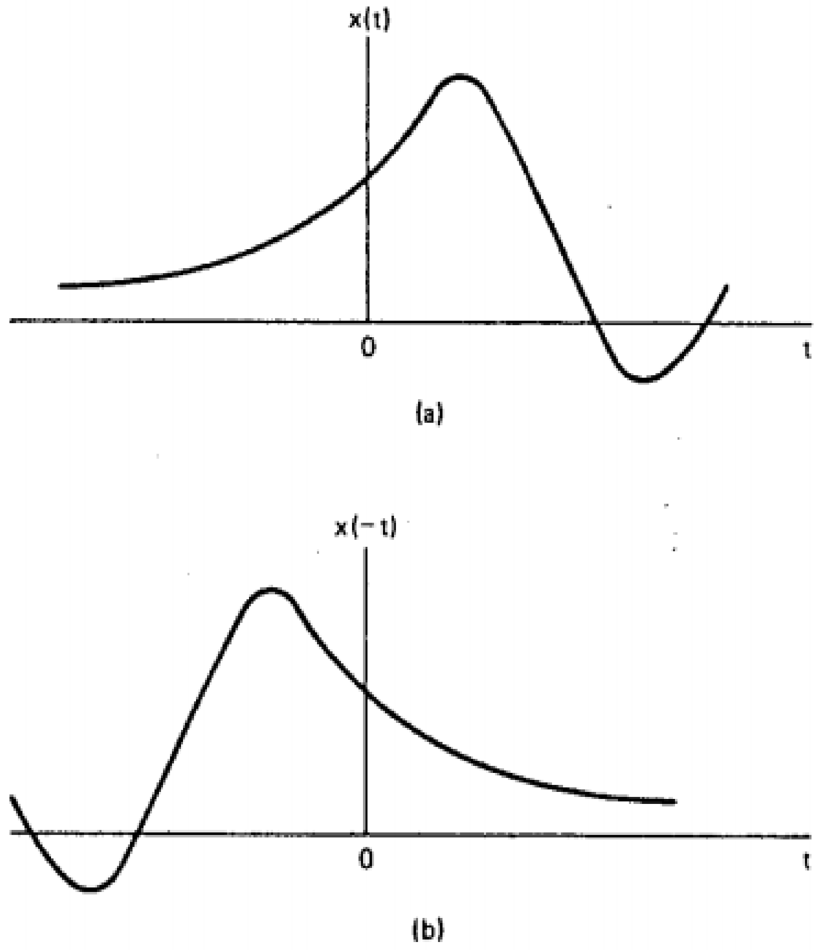


Figure 11 (a) A continuous-time signal $x(t)$; (b) its reflection, $x(-t)$, about $t = 0$.

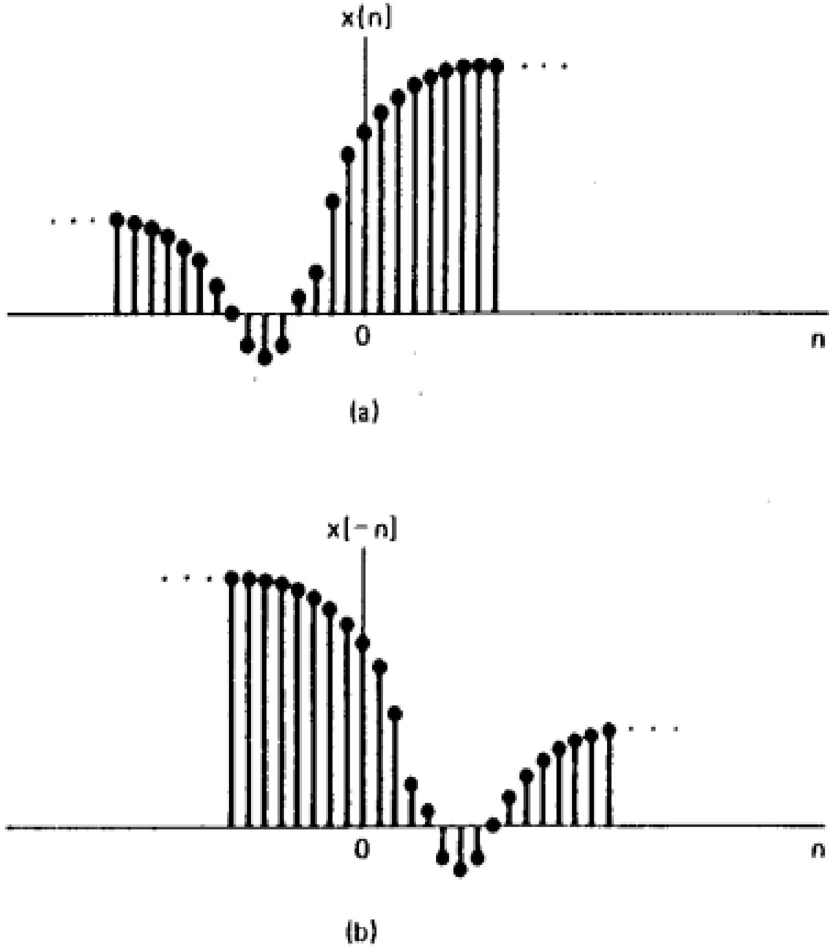


Figure 12 (a) A discrete-time signal $x[n]$; (b) its reflection, $x[-n]$, about $n = 0$.

Cont …

- The following cases are of special interest.
  - **Even Signals:** for which we have $\mathbf{x(-t) = x(t)}$ **for all 't';** that is, ***an even signal is the same as its reflected version.***
  - **Odd Signal:** for which we have $x(-t) = -x(t)$ **for all 't';** that is, ***an odd signal is the negative of its reflected version.***
- ***Similar observation apply to discrete-time(DT) signa***

Example 7:

- Consider the triangular pulse $x(t)$ shown in figure **13(a).**

Find the reflected version of $x(t)$ about the amplitude axis.

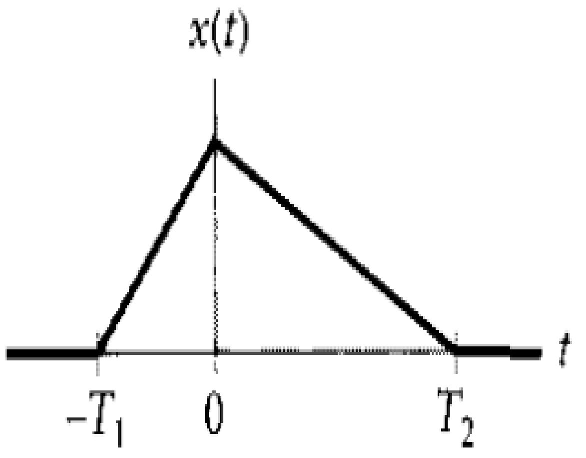


Figure 13(a) Continuous-Time Signal x(t)

Cont …

Solution:

- Replacing the independent variable 't' in $x(t)$, with **'-t'**, we get the result $y(t) = x(-t)$ shown in figure 13(b).

Note that: for this example, we have:

$$x(t) = 0 \; for \; t < -T_1 \; and \; t > T_2$$

Correspondingly, we find that:

$$y(t) = 0 \; for \; t > T_1 \; and \; t < -T_2$$

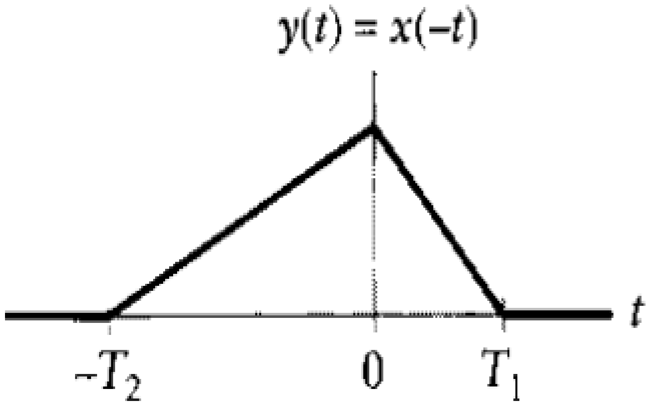


Figure 13(b) Reflected Version of x(t) in figure 13(a) about Amplitude Axis

Example 8:

- Find the composite signal $\boldsymbol{y[n] = x[n] + x[-n]}$ for the discrete-time signal $x[n]$ defined by:

$$i. \quad x[n] = \begin{cases} 1 & n = 1 \\ -1 & n = -1 \\ 0 & n = 0 \; and \; |n| > 1 \end{cases}$$

$$ii. \quad x[n] = \begin{cases} 1 & n = -1 \; and \; n = 1 \\ 0 & n = 0 \; and \; |n| > 1 \end{cases}$$

Answer: for i: $\boldsymbol{y[n]} = 0$ for all integer values of **n.**

for ii: $\boldsymbol{y[n]} = \begin{cases} 2 & n = -1 \; and \; n = 1 \\ 0 & n = 0 \; and \; |n| > 1 \end{cases}$

Cont …

Time Shifting:

- Let $x(t)$ denote a continuous-time signal.
- The time-shifted version of $x(t)$ is defined by:

$$y(t) = x(t - t_o)$$

Where t_o is the time shift.

- If $t_o > 0$, the waveform representing $x(t)$ is **shifted intact/whole to the right,** relative to the time axis.
- If $t_o < 0$, it is ***shifted to the left.***

Example 9:

- ***Figure 14(a)*** shows a rectangular ***pulse*** $\boldsymbol{x(t)}$ of unit amplitude and unit duration. Find $\boldsymbol{y(t) = x(t-2)}$.

Solution:

In this example, the **time shift** $\boldsymbol{t_o}$ equals 2 time units we get the rectangular **pulse** $\boldsymbol{y(t)}$ shown in ***figure 14(b).***

The pulse $\boldsymbol{y(t)}$ has exactly the same shape as the ***original pulse*** $\boldsymbol{x(t)}$***;*** it is merely shifted along the same axis.

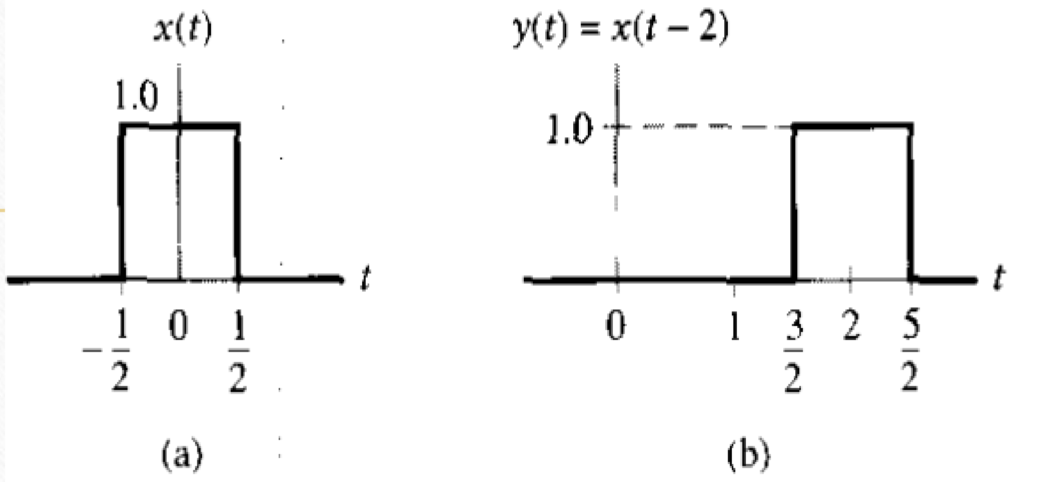


Figure 14: Time-Shifting operation.
(a) Continuous-Time Signal in the form of a Rectangular Pulse of Amplitude 1.0 and Duration 1.0, Symmetric about the origin;
(b) Time-Shifted Version of *x(t)* by 2 time units.

Cont …

- In the case of discrete-time signal $\boldsymbol{x}[\boldsymbol{n}]$, we define its time-shifted version as follows:

$$\boldsymbol{y}[\boldsymbol{n}] = \boldsymbol{x}[\boldsymbol{n} - \boldsymbol{m}]$$

Where the shift m must be an integer; it can be positive or negative.

Example 10:

- Find the time-shifted signal $\mathbf{y[n] = x[n+3]}$ for the discrete-time signal $\mathbf{x[n]}$ defined by:

$$x[n] = \begin{cases} 1 & n = 1, 2 \\ -1 & n = -1, -2 \\ 0 & n = 0, and\ |n| > 2 \end{cases}$$

Answer: $y[n] = \begin{cases} 1, & n = -1, -2 \\ -1, & n = -4, -5 \\ 0, & n = -3, n < -5, \text{ and } n > -1 \end{cases}$

Precedence/Priority Rule For Time Shifting And Time Scaling

- Let $y(t)$ denote a continuous-time signal that is derived from another continuous-time signal $x(t)$ through a combination of time-shifting and time-scaling, as described here:

$$y(t) = x(at - b)$$

- The relationship between $y(t)$ and $x(t)$ satisfies the following conditions:

$$y(0) = x(-b) \qquad \textit{and} \qquad y\left(\frac{b}{a}\right) = x(0)$$

Which provide useful checks on $y(t)$ in terms of corresponding values of $x(t)$.

- To correctly obtain $y(t)$ from $x(t)$, the time-shifting and time-scaling operations must be performed in the correct order.
- The proper order is based on the fact that the scaling operation always replaces "t" by "at", while the time-shifting operation always replaces "t" by "t-b".
- Hence the time-shifting operation is performed first on $x(t)$, resulting in an intermediate signal $v(t)$ defined by:

$$\boldsymbol{v(t) = x(t-b)}$$

- The time-shift has replaced "t" in $x(t)$ by $t - b$.
- Next, the time scaling operation is performed on $v(t)$.
- This replaces "t" by "at", resulting in the desired output:

$$y(t) = v(at) = x(at - b)$$

Example 11:

- Consider the rectangular pulse $x(t)$ of unit amplitude and duration of 2 time units depicted in figure 15. Find $y(t) = x(2t + 3)$.

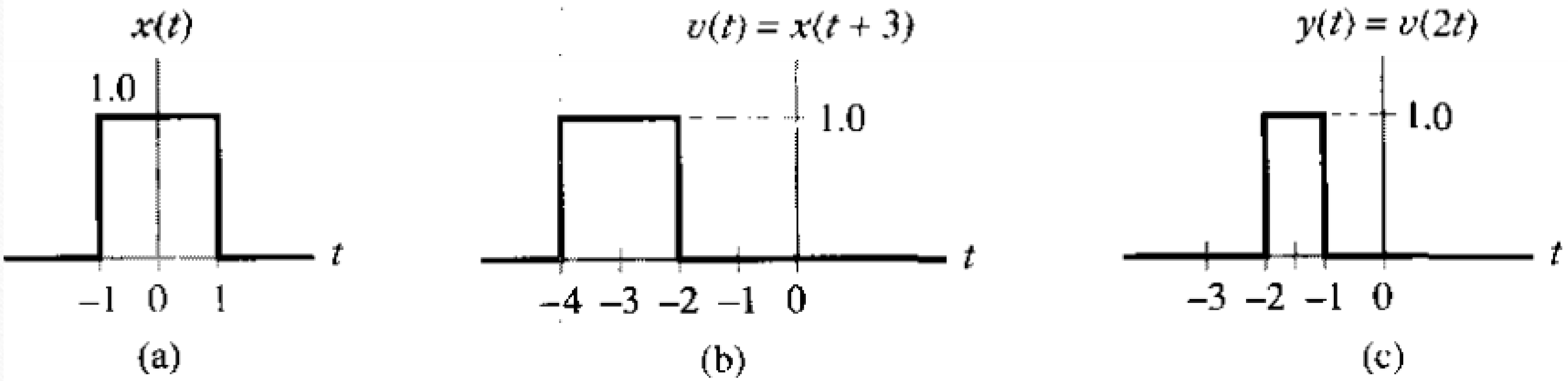


Figure 15: the proper order in which the operations of time scaling and time shifting should be applied for the case of a continuous-time signal. (a) Rectangular pulse $x(t)$ of amplitude 1.0 and duration 2.0, symmetric about the origin (b) Intermediate pulse $v(t)$, representing time-shifted version of $x(t)$ (c) Desired signal $y(t)$, resulting from the compression of $v(t)$ by a factor of 2

Cont …

- Suppose next that we purposely do not follow the precedence rule; that is, we first apply time scaling, followed by time shifting.
- For the given signal $x(t)$, shown in figure 16(a), the wave forms resulting from the application of these two operations are shown in figure 16(b) and (c) respectively.
- The signal $y(t)$ so obtained fails to satisfy the condition of the equation:

$$y(b/a) = x(0)$$

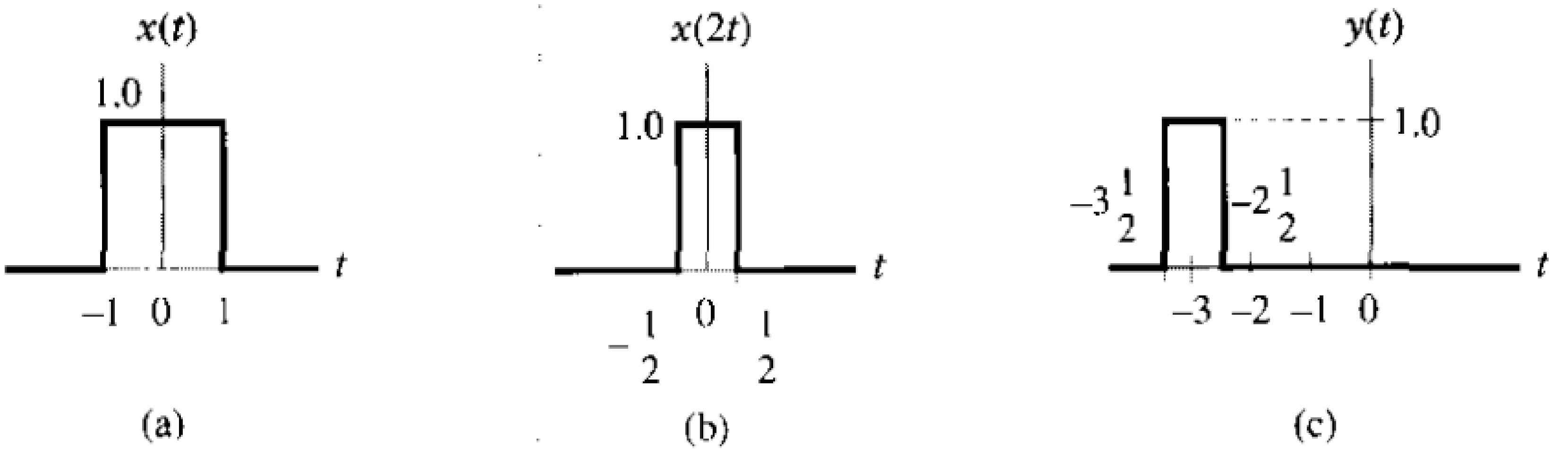


Figure 16 The incorrect way of applying the precedence rule. (a) Signal *x(t)*. (b) Time-Scaled signal *x(2t)* (c) Signal *y(t)* obtained by shifting *x(2t)* by 3 time units.

Example 12:

- Find $y[n] = x[2n+3]$ for a discrete-time signal $x[n]$ defined by:

$$x[n] = \begin{cases} 1 & n = 1, 2 \\ -1 & n = -1, -2 \\ 0 & n = 0, and\ |n| > 2 \end{cases}$$

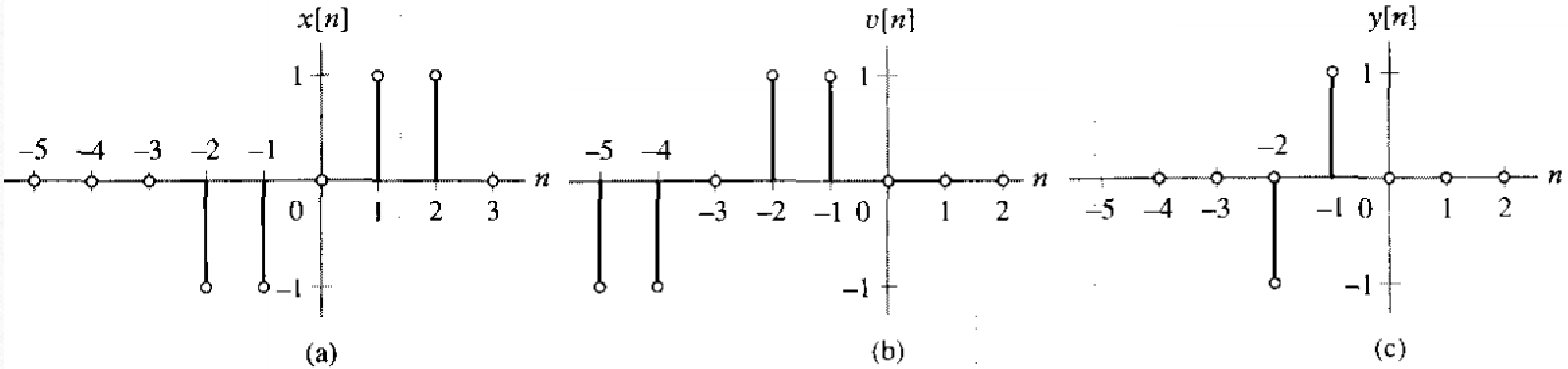


Figure for Example 12

1.3 Elementary Signals

- There are several elementary signals that feature prominently in the study of signals and system.
- The list of elementary signals includes:
  - **Exponential and Sinusoidal Signals**
  - **Step Function**
  - **Impulse Function and**
  - **Ramp Function**
- These elementary signals serve as building blocks for the construction of more complex signals.
- They are also important in their own right, in that they may be used to model physical signals that occur in nature.

Exponential Signals

- A real exponential signal, in its most general form, is written as:

$$x(t) = Be^{at}$$

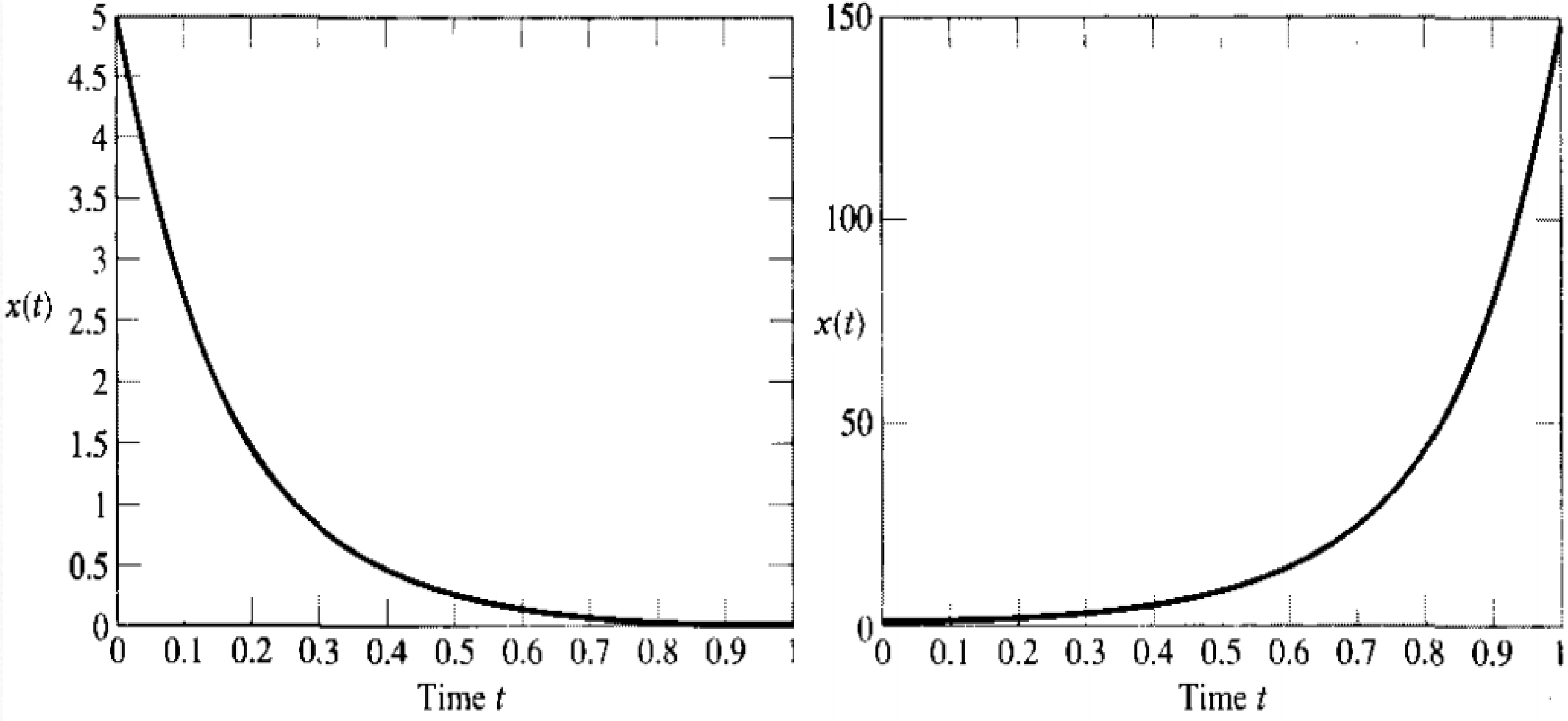
Where both **"B" and "a"** are real parameters.

- The parameter "B" is the amplitude of the exponential signal measured at $t = 0$.
- Depending on weather the other **parameter "a"** is positive or negative, we may identify two special cases:
  - Decaying Exponential, for which $a < 0$
  - Growing Exponential, for which $a > 0$

Cont …

- These two forms of an exponential signal are illustrated in figure 18.
- If $a = 0$, the signal $x(t)$ reduces to a DC signal equal to the constant "B".
- For a physical example of an exponential signal, consider a "lossy" capacitor, as depicted in figure 19.
  - The capacitor has capacitance C, and the loss is represented by shunt resistance R.
  - The capacitor is charged by connecting a battery across it, and then the battery is removed at $t = 0$.
  - Let V_o denote the initial value of the voltage developed across the capacitor.

Cont …



(a) (b)

Figure 18: (a) Decaying Exponential form of Continuous-Time Signal using a = -6 and B = 5.
(b) Growing Exponential form of Continuous-Time Signal using a = 5 and B = 1.

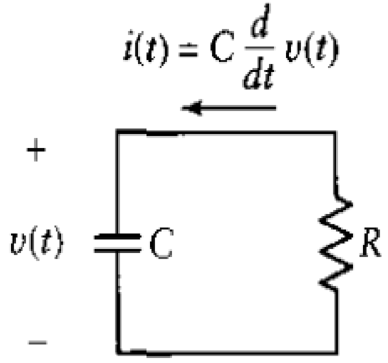


Figure 19: Loosy Capacitor with the loose represented by shunt resistance R

Cont …

- From figure 19 we readily see that the operation of the capacitor for $t \geq 0$ is described by:

$$RC\frac{d}{dt}v(t) + v(t) = 0$$

Where $v(t)$ is the voltage measured across the capacitor at time t.

- The above equation is a differential equation of order one.
- Its solution is given by:

$$v(t) = V_o e^{-t/RC}$$

Where the product term RC plays the role of a time-constant.

Cont …

- These equation shows that the voltage across the capacitor decays exponentially with time at a rate determined by the time constant RC.
  - ➢The larger the resistance R (i.e. the less lossy the capacitor), the slower will be the rate of decay of $v(t)$ with time.
- In discrete-time it is common practice to write a real exponential signal as:

$$x[n] = Br^n$$

- The exponential nature of this signal is readily confirmed by defining:

$$r = e^{\alpha}$$

For some α.

Cont …

- Figure 20 illustrates the decaying and growing forms of a discrete-time exponential signal corresponding to $0 < r < 1$ and $r > 1$, respectively.
- This is where the case of discrete-time exponential signals is distinctly different from continuous-time exponential signals.
- Note that: when $r < 0$, a discrete-time exponential signal assume alternating signs.
- The exponential signals shown in figures 18 and 20 are real valued
- It is possible for an exponential signal to be complex-valued.

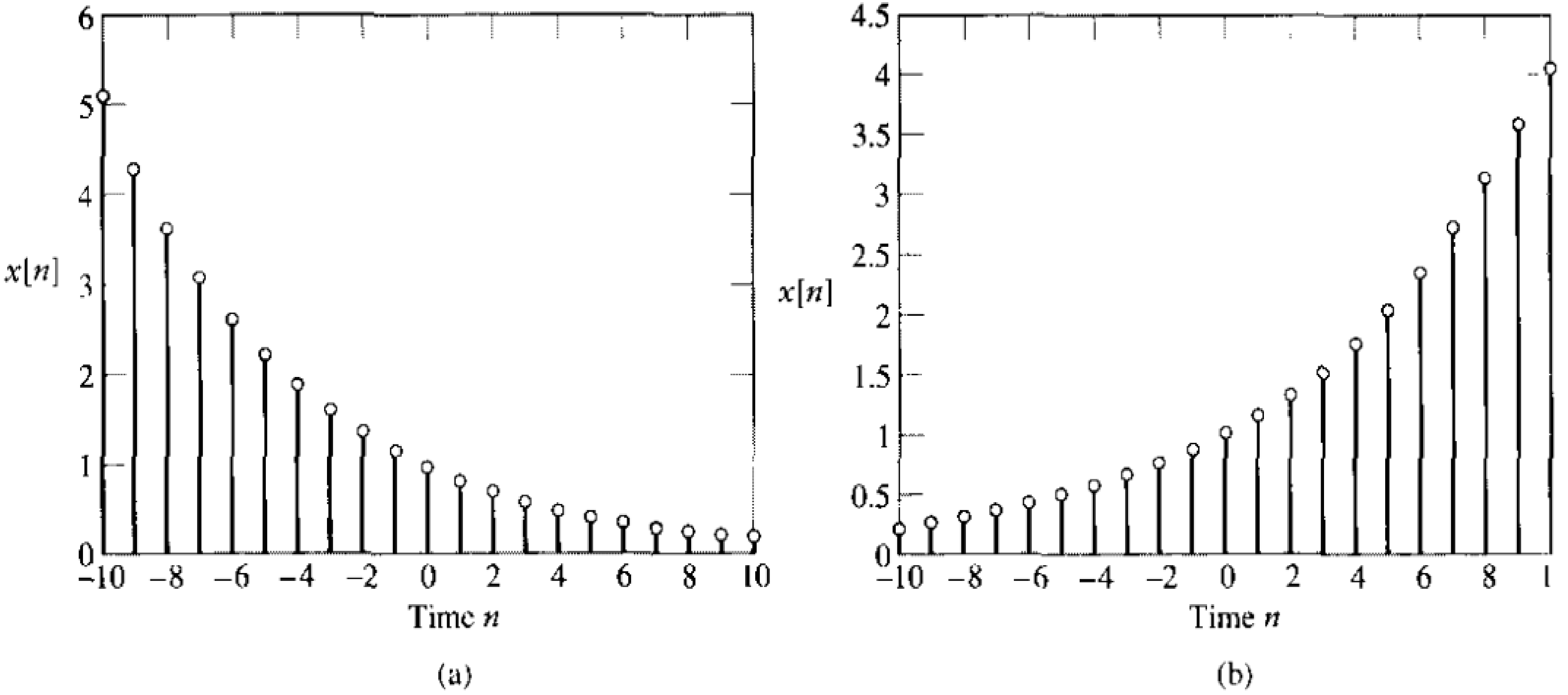


Figure 20: (a) Decaying Exponential form of Discrete-Time Signal
(b) Growing Exponential form of Discrete-Time Signal

Cont …

- The mathematical forms of complex exponential signals are the same as those shown in equations:

$$x(t) = Be^{at} \text{ and } x[n] = Br^{n}$$

With some differences explained here.

- In the continuous-time case, the parameter "B" or parameter "a" or both assume complex values.
- Similarly, in discrete-time case, the parameter "B" or parameter "r" or both assume complex values.
- Two commonly encountered examples of complex exponential signals are $e^{j\omega t}$ and $e^{j\Omega n}$

SINUSOIDAL SIGNAL

- The continuous-time version of a sinusoidal signal, in its most general form, may be written as:

$$x(t) = A * \cos(\omega t + \Phi)$$

Where:

- A is the amplitude
- ω is the frequency in radians per second and
- Φ is the phase angle in radians

- Figure 21(a) presents the wave form of a sinusoidal signal for $A = 4 \; and \; \Phi = +\pi/6$
- A sinusoidal signal is an example of a periodic signal, the period of which is:

$$T = \frac{2\pi}{\omega}$$

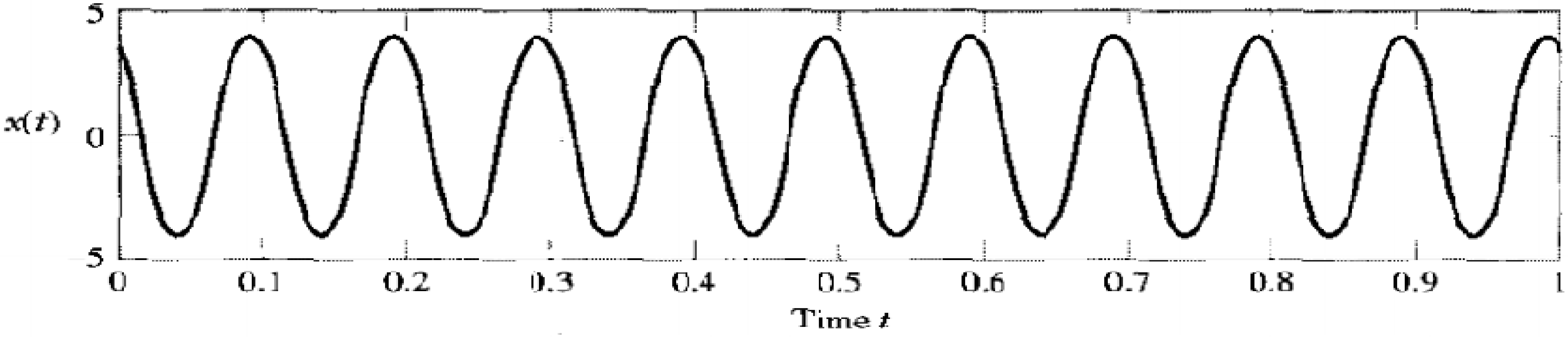


Figure 21(a): Sinusoidal Signal $A*\cos\ (\omega t+\Phi)$ with phase $\Phi=+\ \pi/6$ radians

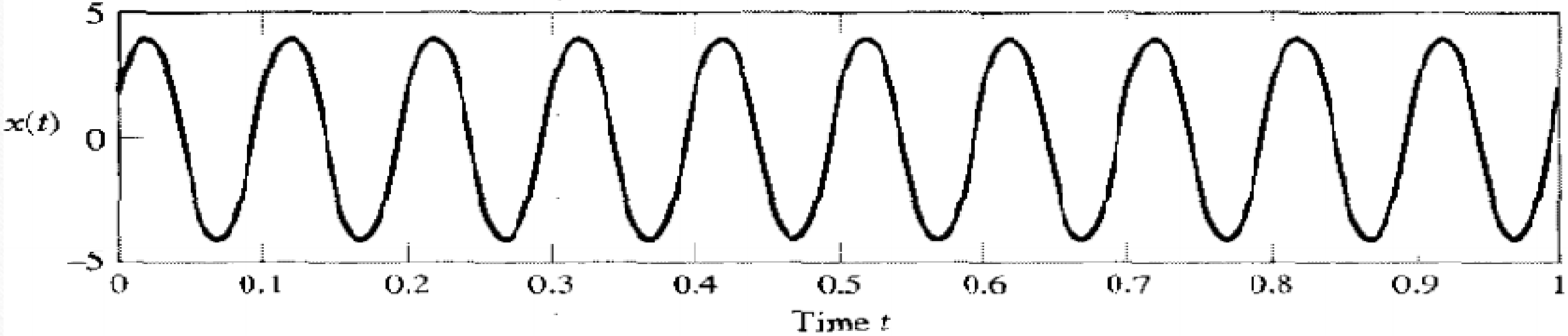


Figure 21(b): Sinusoidal Signal $A*\sin\ (\omega t+\Phi)$ with phase $\Phi=+\ \pi/6$ radians

Cont …

- We may readily prove the periodicity property of a sinusoidal signal by using the above equation to write:

$$
\begin{aligned}
x(t+T) &= A * \cos(\omega(t+T)+\Phi) \\
&= A * \cos(\omega t+\omega T+\Phi) \\
&= A * \cos(\omega t+2\pi+\Phi) \\
&= A * \cos(\omega t+\Phi) \\
&= x(t)
\end{aligned}
$$

Which satisfies the defining condition of a periodic signal.

Cont …

- To illustrate the generation of a sinusoidal signal, consider the circuit of figure 22 consisting of an inductor and capacitor connected in parallel.
- It is assumed that the losses in both components of the circuit are small enough for them to be considered "ideal".
- The voltage developed across the capacitor at time $t = 0$ is equal to V_o.
- The operation of the circuit in figure 22 for $t \geq 0$ is described by:

$$LC\frac{d^2}{dt^2}v(t) + v(t) = 0$$

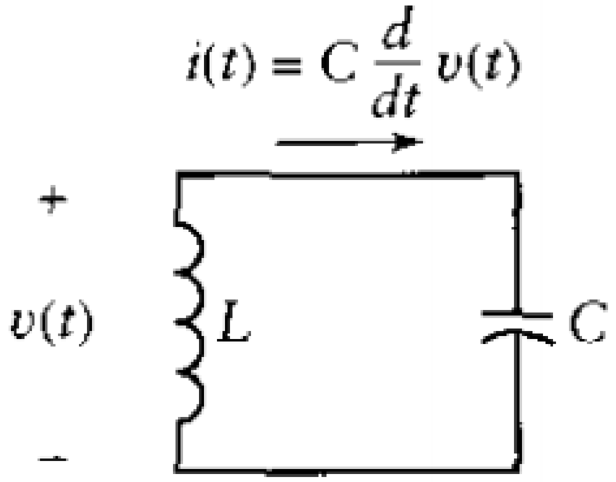


Figure 22: Parallel LC circuit

Where:

Cont …

- $v(t)$ is the voltage across the capacitor at time t.
- C is the capacitance of the capacitor,
- L is the inductance of the inductor

- The above equation is a differential equation of order two.
- Its solution is given by:

$$v(t) = V_o \cos(\omega_o t)\,, \qquad t \geq 0$$

Where ω_o is the natural angular frequency of oscillation of the circuit:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Cont …

- The equation for instantaneous voltage, $v(t) = V_o \cos(\omega_o t)$, describes a sinusoidal signal of amplitude $\mathrm{A} = V_o$, frequency $\omega = \omega_o$ and phase angle $\Phi = 0$.
- Consider next the discrete-time version of a sinusoidal signal, written as:

$$x[n] = A * \cos(\Omega n + \Phi)$$

- The period of a periodic discrete-time signal is measured in samples.
- Thus for $x[n]$ to be periodic with a period of N samples, say, it must satisfy the condition of $x[n] = x[n + N]$ for all integer "n" and some integer "N".

Cont …

- Substituting $n + N$ for n in the above equation yields:

$$x[n+N] = A * \cos(\Omega n + \Omega N + \Phi)$$

For the condition of periodic signal to be satisfied, in general, we require that:

$$\Omega N = 2\pi m \qquad radians$$

$$or \qquad = \frac{2\pi m}{N} \quad \left(\frac{radians}{cycle}\right) \quad integer\ m, N$$

- The important point to note here is that, unlike continuous-time sinusoidal signals, not all discrete-time sinusoidal systems with arbitrary values of Ω are periodic.

- Specifically, for the discrete-time sinusoidal signal described in:

$$x[n] = A * \cos(\Omega n + \Phi)$$

to be periodic, the angular frequency Ω must be a rational multiple of 2π as indicated in:

$$\Omega = \frac{2\pi m}{N}$$

Figure 23 illustrates a discrete-time sinusoidal signal for $A = 1, \Phi = 0 \; and \; N = 12.$

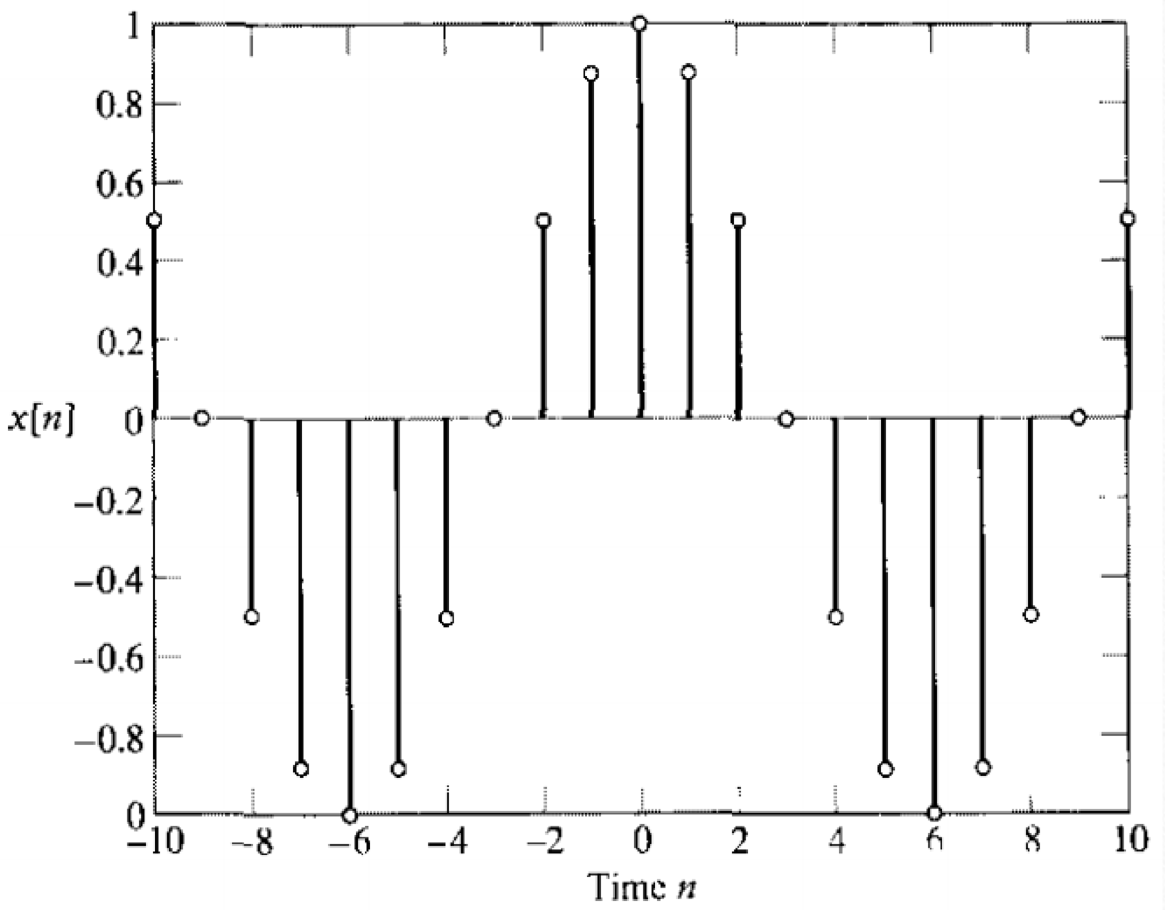


Figure 23: Discrete-Time Sinusoidal Signal

Example 13:

- A pair of sinusoidal signals with a common angular frequency is defined by:

$$x_1[n] = \sin[5\pi n] \qquad and \qquad x_2[n] = \sqrt{3}\cos[5\pi n]$$

a) Specify the condition which the period N of both $x_1[n]$ and $x_2[n]$ must satisfy for them to be periodic.

b) Evaluate the amplitude and phase angle of the composite sinusoidal signal:

$$y[n] = x_1[n] + x_2[n]$$

Solution:

a) The angular frequency of both $x_1[n]$ and $x_2[n]$ is:

$$\Omega = 5\pi \; radians/cycle$$

Cont …

- Solving for the period N:

$$N = \frac{2\pi m}{\Omega} = \frac{2\pi m}{5\pi} = \frac{2m}{5}$$

- For $x_1[n]$ and $x_2[n]$ to be periodic, their period N must be an integer.
- This can only be satisfied for $m = 5, 10, 15, \ldots$ which results $N = 2, 4, 6, \ldots$

b) We wish to express $y[n]$ in the form:

$$y[n] = A * \cos(\Omega n + \Phi)$$

Recall the trigonometric identity:

$$A * \cos(\Omega n + \Phi) = A * \cos(\Omega n) * \cos(\Phi) - A * \sin(\Omega n) * \sin(\Phi)$$

Cont …

- Identifying $\Omega = 5\pi$, we see that the right-hand side of this identity is of the same form as $x_1[n] + x_2[n]$.
- We may therefore write:

$$A sin(\Phi) = -1 \quad and \quad A cos(\Phi) = \sqrt{3}$$

Hence:

$$\tan(\Phi) = \frac{\sin(\Phi)}{cos(\Phi)} = \frac{Amplitude\ of\ x_1[n]}{Amplitude\ of\ x_2[n]} = \frac{-1}{\sqrt{3}}$$

For which we find that $\Phi = -\pi/6$ radians.

Cont …

- Similarly, the amplitude A is given by:

$$A = \sqrt{(Amplitude\ of\ x_1[n])^2 + (Amplitude\ of\ x_2[n])^2} = \sqrt{1+3} = 2$$

Accordingly, we may express $y[n]$ as: $y[n] = 2\cos(5\pi n - \pi/6)$

Example 14: Consider the following sinusoidal signals. Determine whether each $x[n]$ is periodic, and if it is, find its fundamental period.

a) $x[n] = 5\sin[2n]$

b) $x[n] = 5\cos[0.2\pi n]$

c) $x[n] = 5\cos[6\pi n]$

d) $x[n] = 5\sin[6\pi n/35]$

Cont …

<u>Answer</u>:

(a) Non-periodic (b) Periodic, fundamental period = 10. (c) Periodic, fundamental period = 1. (d) Periodic, fundamental period = 35.

RELATION BETWEEN SINUSOIDAL AND COMPLEX EXPONENTIAL SIGNALS

- Consider the complex exponential $e^{j\theta}$.
- Using Euler's identity, we may expand this term as:

$$e^{j\theta} = cos\theta + jsin\theta$$

- This result indicates that, we may express the continuous-time sinusoidal signal of:

$$x(t) = A * \cos(\omega t + \Phi)$$

as the real part of the complex exponential signal:

$$Be^{j\omega t}$$

Cont …

Where B is itself a complex quantity defined by:

$$B = Ae^{j\Phi}$$

- That is, we may write:

$$A * \cos(\omega t + \Phi) = Re\{Be^{j\omega t}\}$$

Where $Re\{\,\}$ denotes the real part of the complex quantity enclosed inside the braces.

- We may readily prove this relation by noting that:

$$\begin{aligned} Be^{j\omega t} &= Ae^{j\Phi}e^{j\omega t} \\ &= Ae^{j(\omega t+\Phi)} \\ &= A\cos(\omega t + \Phi) + jA\sin(\omega t + \Phi) \end{aligned}$$

Cont …

- Previously the sinusoidal signal is defined in terms of a cosine function.
- Of course, we may also define a continuous time sinusoidal signal in terms of a sine function, as shown by:

$$x(t) = A\sin(\omega t + \Phi)$$

- Which is represented by the imaginary part of the complex exponential signal $Be^{j\omega t}$.
- That is, we may write:

$$A\sin(\omega t + \Phi) = Im\{Be^{j\omega t}\}$$

Where $Im\{\ \}$ denotes the imaginary part of the complex quantity enclosed inside the braces.

Cont …

- The sinusoidal signal $A\sin(\omega t+\Phi)$ differs from that of $A\cos(\omega t+\Phi)$ by a phase shift of 90^{o}.
- That is, the sinusoidal signal $A\cos(\omega t+\Phi)$ leads the sinusoidal signal $A\sin(\omega t+\Phi)$, as illustrated in figure 21 for $\Phi=\pi/6$.
- Similarly, in the discrete-time case we may write:

$$Acos(\Omega n+\Phi)=Re\left\{Be^{j\Omega n}\right\} \quad and \quad Asin(\Omega n+\Phi)=Im\left\{Be^{j\Omega n}\right\}$$

Where B is defined in terms of A and Φ by $B=Ae^{j\Phi}$.

Cont …

- Figure 24 shows the two dimensional representation of complex exponential $e^{j\Omega n}$ for $\Omega = \pi/4$ and $n = 0, 1, \cdots, 7$.
- The projection of each value on the real axis is $cos(\Omega n)$, while the projection on the imaginary axis is $sin(\Omega n)$.

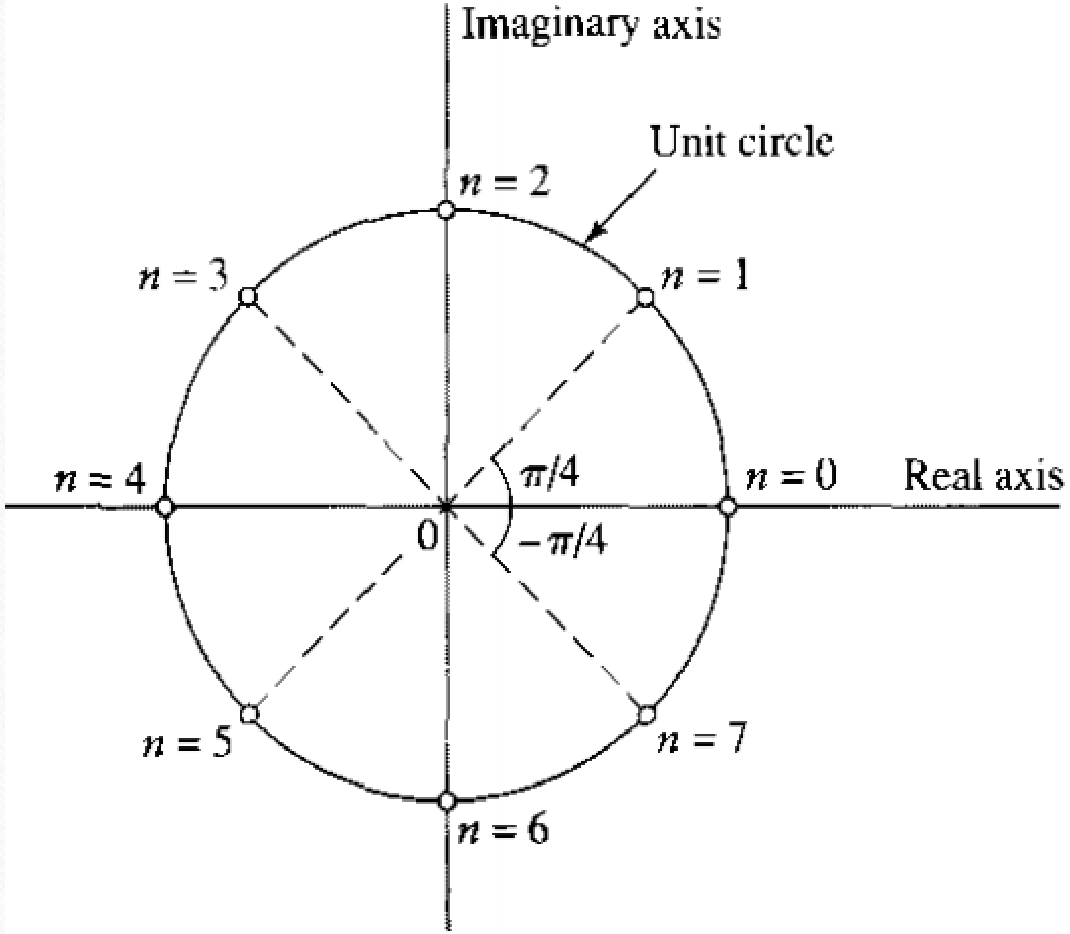


Figure 24: Complex plane, showing eight points uniformly distributed on the unit circle

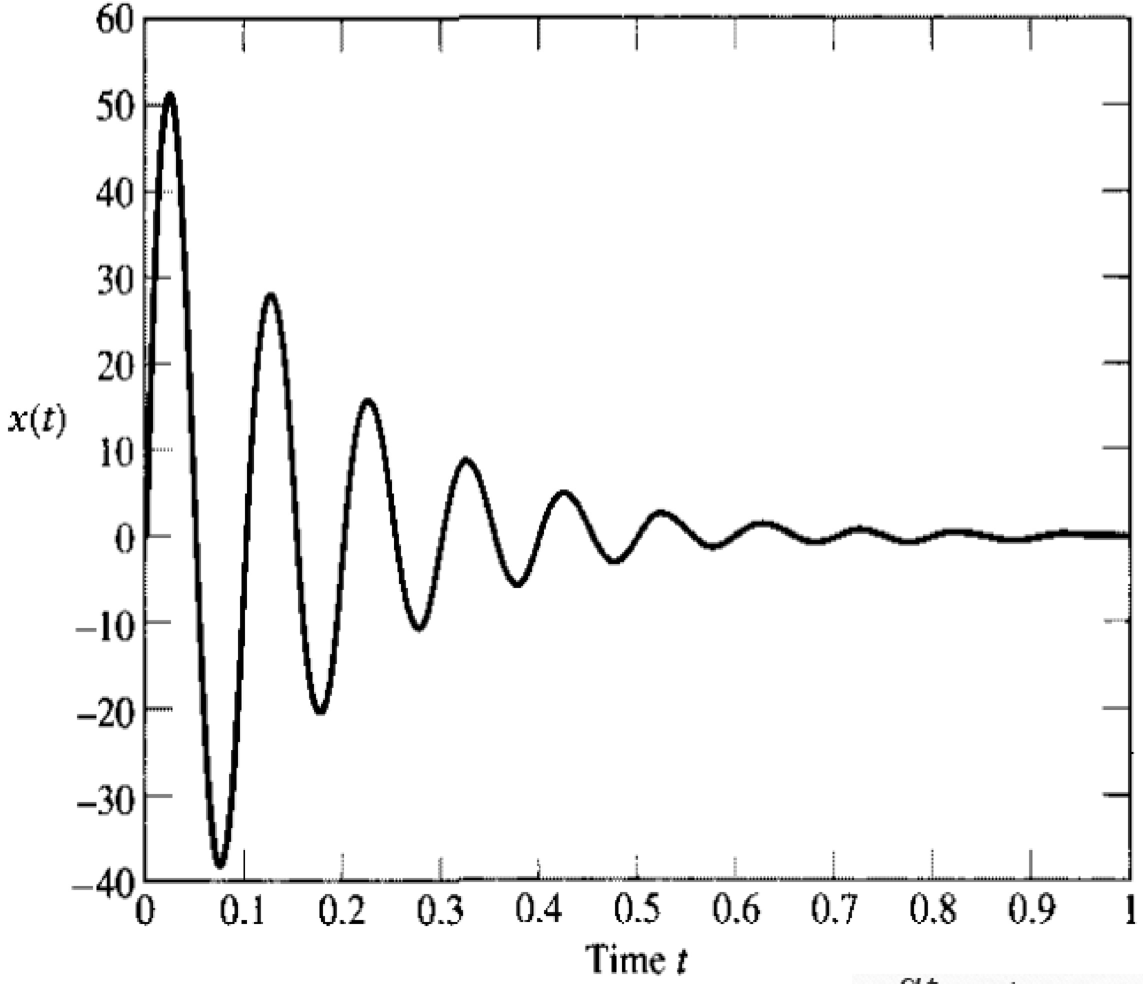


Figure 25: Exponentially damped sinusoidal signal $e^{-\alpha t} sin(\omega t + \Phi)$ **with** $\alpha > 0$

EXPONENTIALLY DAMPED SINUSOIDAL SIGNAL

- The multiplication of a sinusoidal signal by a real-valued decaying exponential signal results in a new signal referred to as an exponentially damped sinusoidal signal.
- Specifically, multiplying the continuous-time sinusoidal signal $Asin(\omega t + \Phi)$ by the exponential $e^{-\alpha t}$ results in the exponentially damped sinusoidal signal:

$$x(t) = Ae^{-\alpha t}sin(\omega t + \Phi), \qquad \alpha > 0$$

- Figure 25 shows the wave form of this signal for $A = 60, \alpha = 6 \; and \; \Phi = 0$.
- For increasing time "t", the amplitude of the sinusoidal oscillations decreases in an exponential fashion, approaching zero for infinite time.

Cont …

- To illustrate the generation of an exponentially damped sinusoidal signal, consider the parallel circuit of figure 26, consisting of a capacitor of capacitance C, an inductor of inductance L, and a resistor of resistance R.
- The resistance R represents the combined effect of losses associated with the inductor and the capacitor.
- Let V_o denote the voltage developed across the capacitor at time $t = 0$.
- The operation of the circuit in figure 26 is described by:

$$C\frac{d}{dt}v(t) + \frac{1}{R}v(t) + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau = 0$$

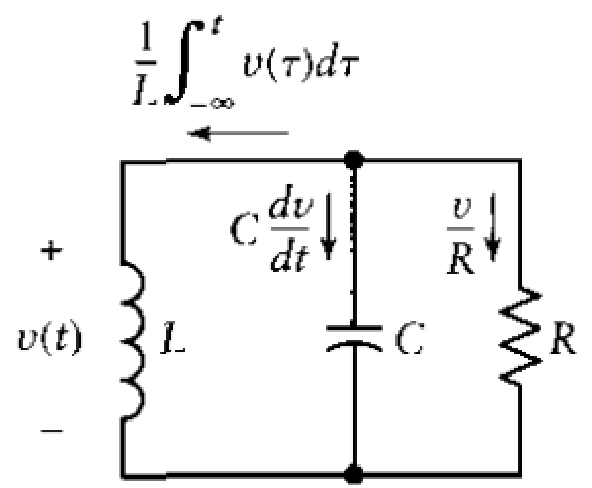


Figure 26: Parallel LCR

Cont …

- Where $v(t)$ is the voltage across the capacitor at time $t \geq 0$.
- The above equation is an integro-differential equation.
- Its solution is given by:

$$v(t) = V_o e^{-t/2RC} \cos(\omega_o t)$$

Where:

$$\omega_o = \sqrt{\frac{1}{LC} - \frac{1}{4C^2R^2}}$$

In the last equation it is assumed that $4CR^2 > L$

Cont …

- Comparing $x(t) = Ae^{-\alpha t} sin(\omega t + \Phi)$ and $v(t) = V_o e^{-t/2RC} \cos(\omega_o t)$; we have:

$$A = V_o, \quad \alpha = 1/2RC, \quad \omega = \omega_o \quad and \quad \Phi = \pi/2.$$

- Returning to the subject matter at hand, the discrete-time version of the exponentially damped sinusoidal signal is described by:

$$x[n] = Br^n \sin[\Omega n + \Phi]$$

For the signal to decay exponentially with time, the parameter r must lie in the range $0 < r < 1$.